2. **Number Patterns, Sequences and Series**

A. Investigating patterns

Patterns have been a topic of interest for humans for thousands of years. Ancient civilisations, such as those of the Egyptians and the Greeks, used patterns in the design of their architecture. In modern times, mathematicians and scientists use patterns, found through experiments and problems, to discover new ideas.

Conjectures are theories about patterns.

For example – Look at this pattern → 2; 4; 8; 16; 32…. What is happening? The next term is always double the previous term. Our conjecture (or theory) is that you have to multiply the previous term by 2 to get the next term. To write this as a formula we can say \( T_n = 2^n \).

So to get to this formula we know that:

- Term 1 = 2
- Term 2 = \( 2 \times 2 = 2^2 \)
- Term 3 = \( 2 \times 2 \times 2 = 2^3 \)

So you can see that the term number is the same as the exponent in each term.

**Activity 2.1:**

*Find a formula or conjecture for each of the patterns shown below, and give the next three terms of each pattern:*

a) 0; 3; 8; 15;......

b) -1; 1; -3; 3; -5; 5;......

c) \( \frac{1}{64}; \frac{1}{32}; \frac{1}{16}; \ldots \)

d) 1; 3; 9; 27;......

e) 0.1; 0.01; 0.101; 0.0101;......

f) \( x^2y; 2x^3y^3; 4x^4y^5; \ldots \)
Using the calculator to solve Sequence and Series calculations:

Use the following keys from the EL-W535HT scientific calculator:

- \( \begin{array}{c}
x^y \\
\end{array} \)

  This key calculates exponential values.

- \( \begin{array}{c}
2nd F \quad x^y \\
\end{array} \)

  These keys together calculate the xth root of y.

- \( \begin{array}{c}
2nd F \\
\end{array} \)

  This key inputs a fraction or improper fraction with numerator and denominator.

- \( \begin{array}{c}
2nd F \quad a/b \\
\end{array} \)

  These keys together input a mixed fraction.

B. Quadratic Sequences

In a linear or “Arithmetic” sequence you have a first common difference – in other words you add or subtract a constant value. For example: 4, 8, 12, 16… and so on. You add a constant difference of 4. This is an example of a first common difference.

In a quadratic sequence you add either an increasing or decreasing amount every time. That amount always increases or decreases by a constant amount – your second common difference. For example: 1, 3, 7, 13, 21… and so on.

\[
\begin{array}{c}
1 \quad 3 \quad 7 \quad 13 \quad 21 \quad \rightarrow \text{sequence} \\
2 \quad 4 \quad 6 \quad 8 \quad \rightarrow \text{first difference} \\
2 \quad 2 \quad 2 \quad \rightarrow \text{second common/constant difference}
\end{array}
\]

The formula for a quadratic sequence is \( T_n = an^2 + bn + c \): where \( T_n \) is your term value and \( n \) is your term position.
To find $a$, $b$ and $c$ you use these three formula’s:

- $a + b + c =$ the first term, in our example $= 1$
- $3a + b =$ the first “first difference”, in our example $= 2$
- $2a =$ the second common difference, in our example $= 2$.

Now solve these equations from the bottom up ($\uparrow$)

So: $2a = 2$

$a = 1$

Then: $3a + b = 2$

$3(1) + b = 2$

$b = 2 - 3$

$b = -1$

And lastly: $a + b + c = 1$

$1 + (-1) + c = 1$

$c = 1$

Now substitute these values into your formula:

$I$ $E$ $B$ students will be familiar with this formula:

$$T_n = T_1 + (n - 1)f + \frac{(n-1)(n-2)s}{2}$$

- $f =$ first difference
- $s =$ second difference

$$\therefore T_n = 1 + (n - 1)(2) + \frac{(n-1)(n-2)}{2}$$

$$\therefore T_n = 1 + 2n - 2 + n^2 - 3n + 2$$

$$\therefore T_n = n^2 - n + 1$$

Now substitute these values into your formula:

$T_n = 1n^2 - 1n + 1$.

To check that our formula works, we choose any position for example, $n = 3$, and we make sure that it equals the value at that position, $T_n = 7$. $T_3 = 1(3)^2 - 1(3) + 1 = 7$

Activity 2.2

1. **Find the formula for the following sequences:**

   a) 2, 5, 10, 17, 26....
   b) 1, 2, 7, 16, 29, 46....
   c) 3, 6, 12, 21, 33....
   d) 1, 3, 10, 22, 39, 61....
   e) 3, 5, 8, 12, 17....
   f) 2, 6, 11, 17, 24, 32....

2. **Given the sequence: 3, 4, 7, 12, 19, 28... Find:**

   a) The seventh and eighth terms
   b) The formula representing the sequence
   c) The $22^{nd}$ term
   d) If $T_n = 199$, find $n$. 
3. Given the sequence: 2, 4, 9, 17...
   a) Continue the sequence for three more terms
   b) Find the formula of the sequence
   c) Find the 13\textsuperscript{th} term
   d) If T\textsubscript{n} = 612, find n.

4. A chicken farmer goes to a market in order to buy chickens. He knows that on each successive day the prices of the chickens go down. On the first day he buys 10 chickens. On the second day he buys 20 chickens. On the third day he buys 36 chickens, on the fourth day he buys 58 chickens. He continues in this pattern until the last day, when he buys 206 chickens.
   a) How many days does the market go on for?
   b) How many chickens does the farmer buy in total?
   c) If the market continued for two weeks, how many chickens would the farmer buy on the last day?
C. Arithmetic Sequence

An ordered list is called a sequence, where $T_1$ is known as term 1, $T_2$ is known as term 2, and $T_3$ is known as term 3, etc. The general term is $T_n$, where ‘$n$’ is a natural number.

$$T_n = a + (n - 1)d$$

Activity 2.3

1. **Given the general term, write down the first 5 terms, as well as the 100$^{th}$ term of the sequence:**

   a) $T_n = n - 3$
   b) $T_n = 7 - n$
   c) $T_n = 2n - 4$
   d) $T_n = 3n + 1$
   e) $T_n = -n + 2$
   f) $T_n = \frac{4}{3}n + 4$

2. **Given the general term, calculate the value of the term indicated:**

   a) $T_n = 4n + 1$, find $T_{12}$
   b) $T_n = 3n + \frac{1}{2}$, find $T_4$

**Example:** Given the following sequence, find the formula:

3; 6; 9; 12; 15; .......

Therefore:

The common difference is $T_2 - T_1 = 6 - 3 = 3$
Now substitute the first term into $a$ and you have your formula:

$$\therefore T_n = 3 + (n - 1)(3)$$
Activity 2.3 continued

3. Determine the 15th and 100th term of each of the following arithmetic sequences by first finding \( T_n \):

   a) 9; 12; 15; …
   b) 0; \(-\frac{5}{2}\); \(-5\); …
   c) 7 + 9x; 8 + 11x; 9 + 13x
   d) \(a = 5; d = 7x\)

4. Determine which term of the arithmetic sequence is equal to the term given in brackets:

   a) -2; 1; 4… (109)
   b) \(\frac{1}{2}; -\frac{3}{7}; -\frac{19}{14}; \ldots\) (-19)
   c) \(x; 2x + 3; 3x + 6; \ldots\) (50x + 147)
   d) \(\log \sqrt{2}; \log 4; \log 8\sqrt{2}; \ldots\) (\( \log 2^{23} \))

5. Determine the arithmetic sequence and the 14th term in each of the following:

   a) The 4th term is 14 and the 20th term is 94.
   b) The 7th term is 12 and the 33rd term is -40.
   c) The 5th term is 2 + 3x and \(d = 1 + x\)
   d) The 6th term is 5x - 2 and \(a = -3\)

6. Consider the following arithmetic sequence:
   \(x + 3; 2x - 2; 5x + 1; \ldots\)

   a) Find the value of \(x\).
   b) Write down the first 3 terms.
   c) Determine the 20th term of this sequence.
   d) Which term in this sequence will be equal -64?

More notes on arithmetic sequences:

An arithmetic sequence, or progression, is any sequence where the same difference occurs between each term within that sequence.

We use the following for the formula: \( T_n = a + (n - 1)d \)

‘a’ is the first term
‘d’ is the common difference between the terms
Activity 2.4

1. Determine the 20\textsuperscript{th} term of the following arithmetic sequences:
   a) 6; 12; 18;... b) -11; -9; -7;...
   c) p + 2q; 3p + 3q; 5p + 4q;... d) a = -\frac{1}{4}, d = \frac{3}{4}
   e) a = 2, T_6 = 62 (HINT: Find ‘d’ first)

2. For each of the following formulas, find:
   (i) Term 1 (a) (ii) Common difference (d)
   a) T_n = 4n – 2 b) T_n = 5 + 3n
   c) T_n = \frac{1}{2}n d) T_n = 6-2n

3. Determine which term in each of the below sequences:
   a) 2; 4; 6; ... is equal to 48
   b) -3; 1; 5; ... is equal to 81
   c) 2\frac{1}{4}; 2\frac{1}{2}; 2\frac{3}{4}; is equal to 6\frac{3}{4}
   d) a; 2a + 1; 3a + 2; ... is equal to 12a + 11

4. Given the following terms of an arithmetic sequence, determine the first 3 terms, and then the value of T_{25}:
   a) T_8 = 22, and T_{17} = 49 b) T_{11} = -28, and T_{30} = -104
   c) T_6 = 10, and T_{21} = 14\frac{1}{2} d) T_4 = 7x + 10, and T_{15} = 29x + 43
D. Geometric Sequences

Notes:

In a geometric sequence, there is a common ratio, ‘r’, which is calculated by dividing any term of a sequence by the previous term in the sequence. Therefore:

\[
\frac{T_{n+1}}{T_n} = r, \text{ for } n \geq 1 \text{ this can also be used to prove that a sequence is geometric:}
\]

If \[
\frac{T_2}{T_1} = \frac{T_3}{T_2}
\]
then the sequence has a common ratio and \(\therefore\) the sequence is geometric.

The formula for finding the general term in a geometric sequence is:

\[
T_n = a r^{n-1}
\]

**Example 1:**

For the sequence given below:

a) show that it is a geometric sequence, and
b) calculate the values for ‘a’ and ‘r’ to find the general term, in order to calculate the next three terms of the sequence.

72; 12; 2;…..

**Answer:**

a) \(T_1 = 72\)

Therefore, \(\frac{T_2}{T_1} = \frac{12}{72} = \frac{1}{6}\) and \(\frac{T_3}{T_2} = \frac{2}{12} = \frac{1}{6}\)

\(T_2 = 12\)

There is a common ratio of \(\frac{1}{6}\) and \(a = 72\)

\(T_3 = 2\)

The general term is: \(T_n = 72 \cdot \left(\frac{1}{6}\right)^{n-1}\)

The next three terms of the sequence are:

\(T_4 = 72 \cdot \left(\frac{1}{6}\right)^{4-1}, \quad T_5 = 72 \cdot \left(\frac{1}{6}\right)^{5-1}, \quad T_6 = 72 \cdot \left(\frac{1}{6}\right)^{6-1}\)

\(= \frac{1}{3}, \quad = \frac{1}{18}, \quad = \frac{1}{108}\)

All of these can be entered straight into your SHARP EL-W535HT calculator by pressing these buttons: 

7 2  ( 2ndF  1 . \(\sqrt{x}\) 6  \(\sqrt{x}\) - 1 ENTER)
Example 2:

Given the following geometric sequence, calculate the 10\textsuperscript{th} and 21\textsuperscript{st} terms:

-3; -9; -27; ....

Answer:

\[ a = -3, \quad r = 3 \text{ (remember } \frac{T_2}{T_1} = \frac{T_3}{T_2}) \]

\[ T_{10} = -3 \times (3)^{10-1} = -59\,049 \text{ or } -3 \times 3^9, \text{ and} \]

\[ T_{21} = -3 \times (3)^{21-1} = -1.046 \times 10^{10} \text{ or } -3 \times 3^{20} \]

You can also put this straight into your EL-W535HT calculator by pressing these buttons:

\[ \begin{align*}
\text{(-)} & \quad 3 \times 3 \quad y^{x} \quad 1 \quad 0 \quad -1 \quad = \\
\end{align*} \]

Example 3:

Given that \( T_3 = 8 \) and \( T_{15} = 32\,768 \), determine the first three terms of the geometric sequence.

\[ T_3 = ar^{3-1} = 8 \quad \text{(1)} \]

\[ T_{15} = ar^{15-1} = 32\,768 \quad \text{(2)} \]

Therefore:

\[ \frac{ar^{14}}{ar^2} = \frac{32\,768}{8} \]

\[ r^{12} = 4,096 \]

\[ r = \sqrt[12]{4,096} \]

\[ \therefore r = 2 \]

Substitute \( r = 2 \) into (1):

\[ 8 = a \cdot (2)^2 \]

\[ 8 = 4a \]

Therefore: \( 2 = a \)

\[ T_1 = 2; \quad T_2 = 2 \times 2 = 4; \quad T_3 = 2 \times 2^2 = 8 \]

Again you can simply put this into your EL-W535HT by pressing these buttons:

\[ \begin{align*}
1 \quad 2 \quad \text{2ndF} \quad y^{x} \quad 4 \quad 0 \quad = \\
9 \quad 6 \quad = \\
\end{align*} \]
Example 4:

In the following geometric sequence \( \frac{1}{3}, \frac{1}{15}, \frac{1}{75} \ldots \) Which term in this sequence is equal to \( \frac{1}{9375} \)?

\[
a = \frac{1}{3} \quad \quad \quad T_n = \frac{1}{3} \times \left(\frac{1}{5}\right)^{n-1} = \frac{1}{9375}
\]
\[
r = \frac{16}{15} = \frac{1}{5}
\]
\[
\therefore \left(\frac{1}{5}\right)^{n-1} = \frac{1}{3125}
\]
\[
\therefore n - 1 = \log_{\frac{1}{5}} \frac{1}{3125}
\]
\[
\therefore n - 1 = 5
\]
\[
\therefore n = 6
\]

Example 5:

Given that \( T_7 = 360 \) and a common ratio of 3, determine the first three terms of the geometric sequence.

\[
T_7 = a \cdot r^{7-1} = 360
\]
\[
a \cdot 3^6 = 360
\]
\[
a \cdot 729 = 360
\]
\[
a = \frac{360}{729}
\]
\[
a = \frac{40}{81}
\]
\[
\therefore T_1 = \frac{40}{81} ; \quad T_2 = \frac{40}{27} ; \quad T_3 = \frac{40}{9}
\]

Activity 2.5

1. Say whether the following sequences are geometric or arithmetic. Find the next three terms in each of the sequences:

   a) \( \frac{1}{4}, \frac{1}{10}, \frac{1}{25} \)  
   b) 3; 12; 48
   c) -7; -10; -13  
   d) 5; 9; 13
   e) \( -\frac{3}{4}, -\frac{9}{8}, -\frac{27}{16} \)  
   f) \( \frac{2}{7}, \frac{11}{14}, \frac{9}{7} \)
2. From the given terms, calculate the common ratio, and then calculate the 21\textsuperscript{st} term of the geometric sequence:

a) \( T_1 = 2 \) and \( T_{12} = \frac{1}{1024} \)

b) \( T_1 = \frac{1}{2} \) and \( T_{12} = -88 \ 573 \frac{1}{2} \)

c) \( T_1 = 4x \) and \( T_{12} = 8192x^{12} \)

d) \( T_1 = 8 \) and \( T_{12} = \frac{177147}{256} \)

3. Given ‘a’ and ‘r’, find the first three terms and also find out which term is equal to the value shown below:

a) \( a = 1, r = 2, T_n = 32 \ 768 \)

b) \( a = \frac{1}{2}, r = 3, T_n = 3280 \ \frac{1}{2} \)

c) \( a = \frac{1}{16}, r = 2, T_n = 4096 \)

d) \( a = 8, r = \frac{1}{4}, T_n = \frac{1}{32} \)

4. Calculate ‘r’, where ‘a’ is given along with a term total:

a) \( a = 2 \) and \( T_5 = 32 \)

b) \( a = 1 \) and \( T_7 = 729 \)

c) \( a = \frac{1}{3} \) and \( T_6 = 2 \ 592 \)

d) \( a = -5 \) and \( T_9 = -\frac{5}{256} \)

5. In a geometric sequence, the first three terms are given as:
\( (p + 2); (p - 2); \) and \( p \). Find the value of \( p \), and hence the first three terms.
E. Series

- A sequence is an ordered list, i.e. \( T_1; T_2; T_3; \ldots; T_n \)
- A series is the sum of the terms of the sequence, i.e. \( T_1 + T_2 + T_3 + \ldots + T_n \)
- A finite series is the sum of a given number of terms, whereas an infinite series is the sum of all the terms of a sequence. An infinite series only occurs when a series is geometric and \(-1 < r < 1; r \neq 0\) (or in other words your ratio is a positive or negative fraction). This concept will be explored in more detail in Section F page 55.
- The formula for the sum of ‘n’ terms of an arithmetic series:
  \[ S_n = \frac{n}{2} [2a + (n - 1)d] \]
- If the last term of the sequence is given, then the formula is as follows:
  \[ S_n = \frac{n}{2} [a + l], l = a + (n - 1)d \]
- \( S_n = S_{n-1} + T_n, \therefore T_n = S_n - S_{n-1} \)
- The formula for the sum of a geometric series to n terms is:
  \[ S_n = \frac{a(1-r^n)}{1-r}, r \neq 1 \]

As a matric student you need to be able to prove these two series (sum) formulae. Here is how to do it:

\[ S_n = a + [a + d] + [a + 2d] + [a + 3d] + \ldots + [a + (n - 2)d] + [a + (n - 1)d] \]
\[ S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \ldots + [a + d] + a \]

Add these two sums together:
\[ 2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d] \]
\[ = n[2a + (n - 1)d] \]
\[ \therefore S_n = \frac{n}{2} [2a + (n - 1)d] \]

And for the geometric series the proof is just as easy:

\[ S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \]
\[ r \times S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^{n} \quad \text{(Multiply by } r) \]
\[ rS_n - S_n = -a + 0 + 0 + \ldots + 0 + 0 + ar^n \]
\[ \therefore rS_n - S_n = ar^n - a \]
\[ \therefore S_n(r - 1) = a(r^n - 1) \]
\[ \therefore S_n = \frac{a(r^n - 1)}{(r-1)}, r \neq 1. \]
Example 1:

If \( T_1 = 2 \) and \( d = 5 \),

a) Determine \( T_1 \) to \( T_5 \) of the sequence and 
b) Calculate the sum of these first five terms.

Answers

a) \( T_1 = 2, T_2 = 7, T_3 = 12, T_4 = 17, T_5 = 22 \)

b) \( S_n = \frac{n}{2} \left[ a + l \right] \)

\( S_5 = \frac{5}{2} [2 + 22] \)

\( S_5 = 60 \)

Example 2:

Given that \( S_n = \frac{1}{4} n^2 - n \) find the first 5 terms and say whether the sequence is arithmetic or geometric, and then calculate the sum of these 5 terms.

\[
\begin{align*}
S_1 &= \frac{1}{4} (1)^2 - (1) = -\frac{3}{4} \quad \therefore T_1 = S_1 = -\frac{3}{4} \\
S_2 &= \frac{1}{4} (2)^2 - (2) = -1 \quad \therefore T_2 = S_2 - S_1 = -1 - \left( -\frac{3}{4} \right) = -\frac{1}{4}S_3 = \\
S_3 &= \frac{1}{4} (3)^2 - (3) = -\frac{3}{4} \quad \therefore T_3 = S_3 - S_2 = \left( -\frac{3}{4} \right) - (-1) = \frac{1}{4} \\
S_4 &= \frac{1}{4} (4)^2 - (4) = 0 \quad \therefore T_4 = S_4 - S_3 = 0 - \left( -\frac{3}{4} \right) = \frac{3}{4} \\
S_5 &= \frac{1}{4} (5)^2 - (5) = 1 \frac{1}{4} \quad \therefore T_5 = S_5 - S_4 = 1 \frac{1}{4} - 0 = 1 \frac{1}{4} 
\end{align*}
\]

\[ d = \frac{1}{2} \] so this is an arithmetic sequence as there is a common difference.

\[ \therefore S_5 = \frac{5}{2} \left[ 2 \left( -\frac{3}{4} \right) + (5 - 1) \left( \frac{1}{2} \right) \right] \]

\[ S_5 = 1 \frac{3}{4} \]

You can put this sequence straight into your SHARP EL-W535 calculator and get the correct answer:

\[
\begin{align*}
&\text{a/b} & 5 & \downarrow & 2 & \rightarrow & ( & 2 & ( & 1 & ) & \rightarrow & \text{a/b} & 3 \\
&\downarrow & 4 & \rightarrow & ) & + & ( & 5 & - & 1 & ) & ( & 1 \\
&\text{a/b} & 2 & \rightarrow & ) & \rightarrow & \text{ENTER} & 
\end{align*}
\]
Example 3:

Given that $T_1 = 0.1$ and $T_{15} = 2.9$, calculate the sum of the arithmetic series to 15 terms.

$$T_1 = 0.1 = a$$

$$T_{15} = a + (n - 1)d = 2.9$$

$$0.1 + (14)d = 2.9$$

$$14d = 2.8$$

$$d = \frac{1}{5} \text{ or } 0.2$$

$$\therefore S_n = \frac{n}{2} [2a + (n - 1)d]$$

$$S_{15} = \frac{15}{2} [2(0.1) + 14 \left( \frac{1}{5} \right)]$$

$$S_{15} = 22 \frac{1}{2} \text{ or } 22.5$$

Example 4:

Given an arithmetic sequence of $32 + 28 + 24 + \ldots \ldots$ determine the value of ‘n’ for which the series total is 140.

$$\therefore S_n = \frac{n}{2} [2a + (n - 1)d]$$

$$140 = \frac{n}{2} [2(32) + (n - 1)(-4)]$$

$$280 = n[64 - 4n + 4]$$

$$280 = 64n - 4n^2 + 4n$$

Both values are positive and if they are substituted into the formula

$$0 = 4n^2 - 68n + 280$$

both give a result of 140, so they both satisfy the equation.

$$n = 7 \text{ or } n = 10.$$ 

Example 5:

The sum of the first 10 terms of an arithmetic series is 80. The sum of term 3 and term 7 is 12. Calculate ‘a’ and ‘d’, and hence write down the series.

$$T_3 = a + 2d$$

$$T_7 = a + 6d$$

$$\therefore a + 2d + a + 6d = 12$$

$$2a = 12 - 8d$$

Remember that $2a = 12 - 8d$

$$\therefore S_{10} = 80 = \frac{10}{2} [2a + 9d]$$

$$80 = 5[12 - 8d + 9d]$$

$$80 = 5[12 + d]$$

$$\frac{80}{5} = 12 + d$$

$$16 - 12 = d$$

$$d = 4$$
∴ \[2a = 12 - 8d\]
∴ \[a = 6 - 4d\]
∴ \[a = 6 - 4(4)\]
∴ \[a = -10\]

∴ \[-10 - 6 - 2 + 2 + 6 + \cdots\]

**Example 6:**

Calculate the geometric series: 1+3+9+27.....to 12 terms.

\[S_n = \frac{a(1-r^n)}{1-r}\]
∴ \[S_{12} = \frac{1(1-3^{12})}{1-3}\]
∴ \[S_{12} = 265,720\]

**Example 7:**

Calculate the value of ‘n’, for which the series is equal to \[16\ \frac{383}{2}\], and where term 1 is \[\frac{1}{2}\] and \[r = 2\].

\[S_n = \frac{a(1-r^n)}{1-r}\]
∴ \[16 \ \frac{383}{2} = \frac{\frac{1}{2}(1-2^n)}{1-2}\]
∴ \[-32,767 = \frac{1}{2}(1 - 2^n)\]
∴ \[-32,767 = 1 - 2^n\]
∴ \[32,768 = 2^n\]
∴ \[\log_2 32,768 = n\]
∴ \[n = 15\]

**Activity 2.6**

1. **Given the following general terms, determine the first 3 terms of the sequence, and hence calculate the sum of those three terms.**
   a) \[T_n = -2n\]  
b) \[T_n = 4 - n\]

2. Given the general term, \[T_n = 2n + 3\], calculate the sum of the first 10 terms, i.e. \[S_{10}\].
3. If \( S_{10} = 120 \) and \( S_{11} = 144 \), find the value of \( T_{11} \).

4. If \( S_n = 3n^2 + n \), find the first 5 terms of the sequence, and say what type of sequence it is.

5. Given the following arithmetic series, find the sum: (Hint find \( n \) first)
   
   a) \( 3 + 6 + 9 + 12 + \ldots + 36 \).
   
   b) \( 2 + 2 \frac{1}{2} + 3 + 3 \frac{1}{2} + \ldots + 10 \).
   
   c) \( 7 + 6.75 + 6.5 + 6.25 + \ldots + 3.75 \).

6. Find the number of terms in the arithmetic series if the sum is 45, \( T_2 = \frac{3}{2} \) and \( T_5 = 3 \).

7. \( S_6 = 159 \), \( T_8 - T_5 = 15 \). Find \( S_{10} \).

8. Calculate the sum of each geometric series given below:
   
   a) \( 8 + 4 + 2 + \ldots \ldots \) to 10 terms
   
   b) \( 2 + -4 + 8 + \ldots \ldots \) to 6 terms
   
   c) \( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \ldots \) to 8 terms.

9. If:
   
   a) \( T_1 = 3 \) and \( T_7 = \frac{3}{64} \). Find \( S_{10} \) of the geometric series.
   
   b) \( T_1 = 2 \) and \( T_8 = 4374 \). Find \( S_{15} \) of the geometric series.

10. \( S_6 = 504 \) and \( r = 2 \). Find \( T_1 \).
F. Sigma Notation

\[ \sum_{n=1}^{k} T_n \]

- \( \sum \rightarrow \) This symbol is the Greek capital letter for \( S \). It represents the sum of a number of terms in a sequence. The above example reads as follows: Sigma \( T_n \), starting at \( n = 1 \), and ending with \( k = \) what? (the sum of a number of terms of a general term)

There are certain sigma series that have a particular answer:
- \( \sum_{n=1}^{k} 1 = n \) in other words \( 1 + 1 + 1 + 1 \) and so on.... This is simply counting and so will equal the number of terms in the sequence.
- \( \sum_{n=1}^{k} n = \frac{k(k+1)}{2} \) in other words \( 1 + 2 + 3 + 4 \) and so on... if you use the formula you should find the sum.

For example if you have: \( \sum_{n=1}^{5} n \) then you are saying \( 1 + 2 + 3 + 4 + 5 = 15 \). Simple if you have only 5 terms.
But when you have: \( \sum_{n=1}^{100} n \) then it becomes much more difficult and you need to use the formula \( \frac{k(k+1)}{2} \)

(\( 100(100+1) = 5050 \) which you can put straight into your EL-W535HT.)

**Example 1:** \( \sum_{n=1}^{8} n(n - 1) \)

This example is asking that you calculate the sum where 1 is the first value to be substituted and 8 is the last value to be substituted. So in all, 8 terms will be added to get the answer.

**Solution:**
\[ \sum_{n=1}^{8} n(n - 1) = 1(0) + 2(1) + 3(2) + 4(3) + 5(4) + 6(5) + 7(6) + 8(7) \]
\[ = 0 + 2 + 6 + 12 + 20 + 30 + 42 + 56 \]
\[ = 168 \]

**Example 2:** \( \sum_{n=4}^{12} 8 \)

In this example, no general term is given, so that means that each term in the sequence will be the number 8. There will be 9 terms of 8 added together as the starting value is 4 and the ending value is 12
Solution: \( \sum_{n=4}^{12} 8 \)
\[ = 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 \]
\[ = 72 \]

OR \[ 8 \times 9 = 72 \]

Example 3: In this example, a series is given, which needs to be converted back to sigma notation.

\[ 3 + 6 + 9 + 12 + 15 + 18 + 21 \]
_convert it to sigma notation_

Solution: Try to find the arithmetic or geometric rule for the sequence.

\[ 3 + 6 + 9 + 12 + 15 + 18 + 21 \]
\[ \therefore a = 3; \quad d = 3 \]
\[ \therefore T_n = 3 + (n - 1)(3) \]
\[ \quad = 3 + 3n - 3 \]
\[ \therefore T_n = 3n \]

\[ 3 + 6 + 9 + 12 + 15 + 18 + 21 = \sum_{n=1}^{7} 3n \]

What term are you starting at?

Example 4:

\[ 5 + 9 + 13 + 17 \text{....to k terms} \] is equal to 324. Find k.

Solution:
\[ a = 5, \quad d = 4 \]
\[ S_n = \frac{n}{2} [2a + (n - 1)d] \]
\[ \therefore 324 = \frac{k}{2} [2(5) + (k - 1)4] \]
\[ \therefore 648 = k[10 + 4k - 4] \]
\[ \therefore 0 = 10k + 4k^2 - 4k - 648 \]
\[ \therefore 0 = 4k^2 + 6k - 648 \]
\[ \therefore 0 = 2k^2 + 3k - 324 \]
\[ \therefore 0 = (k - 12)(2k + 27) \]
\[ \therefore k = 12 \quad \text{or} \quad k \neq -\frac{27}{2} \]

as \( n \) is not a fraction or a negative number.
Example 5: \( \sum_{n=1}^{150} (2n + 1) \) If the number of terms given is too large to calculate the sum of, as in the above examples, the formula for arithmetic series or geometric series can be used. First you would need to decide whether the sequence has a common difference or ratio.

Answer:

3 + 5 + 7 + \ldots + 301. \( a = 3, d = 2 \), therefore this an arithmetic sequence. The last term given is 301.

\[ S_n = \frac{n}{2}[2a + (n - 1)d] \quad \text{OR} \quad S_n = \frac{n}{2}(a + l) \]

\[ S_{150} = \frac{150}{2}[2(3) + (150 - 1)(2)] \quad \therefore S_{150} = \frac{150}{2}(3 + 301) \]

\[ = 75[6 + 298] = 75(304) = 22,800 \]

The general term for a geometric sequence is \( T_n = ar^{n-1} \), and the equation for the sum of a geometric series is \( S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1 \). In sigma notation this would read as \( S_n = \sum_{n=1}^{k} ar^{n-1} \).

Example 6:

Given:

\[ S_{100} = \sum_{n=1}^{100} \frac{1}{2}(2)^{n-1} \]

Find the sum of the first 100 terms of the geometric sequence.

Therefore:

\[ a = \frac{1}{2}(2)^0 = \frac{1}{2}; \quad T_2 = \frac{1}{2}(2)^1 = 1; \quad T_3 = \frac{1}{2}(2)^2 = 2; \quad \therefore r = 2 \]

\[ S_n = \frac{a(1-r^n)}{1-r} \]

\[ S_{100} = \frac{\frac{1}{2}(2^{100}-1)}{2-1} \]

\[ \therefore S_{100} = 6.338 \times 10^{29} \]

To put this into your SHARP EL-W535HT press:
Example 7:
Given:
1 + 3 + 9 + 27 + ...... to 12 terms. Calculate the sum of this geometric series.

Answer:
\[ a = 1 \]
\[ r = 3 \]
\[ S_n = \frac{a(1-r^n)}{1-r} \]
\[ S_{12} = \frac{1(1-3^{12})}{1-3} \]
\[ \therefore S_{12} = 265720 \]

Activity 2.7

1. Evaluate (means calculate) the following series by first determining if the series is arithmetic or geometric:
   a) 2 – 6 – 18........... To 8 terms.
   b) 3 + 6 + 9........... To 8 terms.
   c) \( \frac{1}{2} + 1 + 2........... \) To 8 terms.
   d) 16 + 12 + 8........... To 8 terms.

2. Determine \( n \) in each of the following series by first identifying whether it is arithmetic or geometric:
   a) \[ \sum_{k=1}^{n} \left( \frac{1}{2} \right) (3)^{k-1} = 4920 \frac{1}{2} \]
   b) \[ \sum_{k=1}^{n} (2k + 3) = 285 \]
   c) \[ \sum_{k=1}^{n} (5)(3)^{k-1} = 16400 \]

3. Calculate:
   a) \[ \sum_{n=1}^{5} \left( \frac{1}{2} \right) (4)^{n-1} \]
   b) \[ \sum_{n=3}^{8} \left( \frac{5n}{2} \right) \]
   c) \[ \sum_{n=6}^{18} 3.2n \]
   d) \[ \sum_{n=2}^{6} (4) \left( \frac{1}{2} \right)^{n-1} \]

4. Given the following information, find the value of ‘\( n \)’, or ‘\( k \)’:
   a) \[ S_n = 182, a = 8, \text{ and } d = 6 \]
   b) \[ S_n = 3069, a = 3 \text{ and } r = 2. \]
5. **Evaluate the following:**

   a) \( \sum_{n=1}^{100} (3n - 5) \)

   b) \( \sum_{n=1}^{150} \left( \frac{1}{2} \right)^n \)

6. **Write the following in sigma notation; all to \( k \) terms.**

   a) - 1 – 3 – 9.................

   b) 2 + 1 + \( \frac{1}{2} \) .................

**G. Sum to Infinity**

When no last term of a series is given, we cannot calculate a definite sum. We say we are finding the sum to infinity. This is symbolised by using \( \infty \).

In an infinite arithmetic series, the larger the value for ‘\( n \)’, the larger the answer. That is, this series will always diverge \( \rightarrow \) in other words, the answer for a sum to infinity of a divergent series is infinity.

In an infinite geometric series, the result can either be divergent or convergent, depending on the size of ‘\( r \)’. For the series to be divergent, the value for ‘\( r \)’ must be less than -1, \( r = 1 \), or \( r \) is greater than 1. For a convergent series, ‘\( r \)’ must lie between -1 and 1 (remember that \( r \neq 0 \)).

\[
S_\infty = \frac{a}{1-r} \text{ or } \sum_{n=1}^{\infty} (a)(r)^{n-1} = \frac{a}{1-r} \quad \text{because } r^n \rightarrow 0 \text{ as } n \rightarrow \infty
\]

**Example 1:**

Find \( S_\infty \) for the following sequence. Give a reason for the existence or non-existence of \( S_\infty \).

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n
\]

\[ \therefore T_1 = \frac{1}{2}; \quad T_2 = \frac{1}{4}; \quad T_3 = \frac{1}{8} \]

\[ \therefore a = \frac{1}{2} \text{ and } r = \frac{1}{2} \]

\[ \therefore S_\infty = \frac{a}{1-r} \]

\[ \therefore S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \]

\[ \therefore \text{The sum to infinity exists as } -1 < r < 1 \text{ (the sum converges to 1 as } 'n' \text{ increases).} \]
Example 2:

Calculate the following if it exists:

$$\sum_{n=1}^{\infty} \left(2 \left(\frac{3}{2}\right)^{n-1}\right)$$

\[
\therefore a = 2 \quad \text{and} \quad r = \frac{3}{2}
\]

\[
\therefore S_{\infty} \text{ cannot be calculated.}
\]

The sum of this series does not exist as \( r > 1 \)

Example 3:

If \( S_{\infty} \) is equal to \( 10 \frac{2}{3} \) and \( a = 8 \), find the value of ‘\( r \)’:

\[
\therefore S_{\infty} = \frac{a}{1-r}
\]

\[
\therefore 10 \frac{2}{3} = \frac{8}{1-r} \quad \text{Change } 10 \frac{2}{3} \text{ to an improper fraction}
\]

\[
\therefore 32 = \frac{8}{1-r} \quad \text{Cross multiply}
\]

\[
\therefore 32 - 32r = 24
\]

\[
\therefore -32r = -8
\]

\[
\therefore r = \frac{1}{4}
\]

Example 4:

Given the series: \( (p+2) + (p+2)^2 + (p+2)^3 \) ....

Determine for which values of ‘\( p \)’ the series will converge, and hence find the sum of the series in terms of \( p \). (For a converging series, \(-1 < r < 1\))

\[
a = (p+2)
\]

Therefore: \(-1 < p+2 < 1\)

\[
r = (p+2)
\]

\(-3 < p < -1 \) (subtract 2 from both sides)

So for \(-3 < p < -1\), \( S_{\infty} = \frac{p+2}{1-(p+2)} \)

\[
= \frac{p+2}{-1-p} \quad \text{for a convergent series}
\]

Activity 2.8

1. Given the following geometric series, find the sum to infinity:

   a) \( \sum_{n=1}^{\infty} (2) \left(\frac{1}{3}\right)^{n-1} \)
   
   b) \( 5,25 + 0,0525 + 0,000525 \)

   c) \( 1 + \frac{1}{3} + \frac{1}{9} \ldots \)

   d) \( \sum_{p=0}^{\infty} \left(-\frac{1}{2}\right)^p \)
2. Determine whether it is possible to calculate the sum to infinity:

\[ \sum_{n=1}^{\infty} \left( \frac{2}{3} \right) (2)^{n-1} \]

3. If \( S_\infty = 15, \ a = 3 \). Find \( r \).

4. If \( S_\infty = \frac{7}{16}, \ r = \frac{1}{8} \). Find \( a \).

5. A mountain climber is climbing a mountain. In his first hour he climbs 2km. For every hour he climbs thereafter, he is only able to climb \( \frac{3}{5} \) of the previous height, due to an increase in the incline of the mountain. How high is he able to climb before he needs to start using climbing equipment?
H. Problem solving

Steps to solving problems:
- Know what you are being asked to find
- Write down information that will be useful in solving the problem
- Use diagrams to help understand the information in the question
- Make sure that all units given are the same, otherwise convert
- Use correct formulae to assign variables
- If solving a sequence problem, make sure you identify whether it is arithmetic or geometric
- Solve for the unknown value by using appropriate equations
- Make sure the answer is logical and correct. Substitute it back to check the accuracy.

Example:

Matchsticks are arranged in piles. The first pile consists of 10 matchsticks. Each pile thereafter consists of 8 matchsticks more than the previous pile.

How many matchsticks are in the 25th pile?
If a pile consists of 98 matches, work out which pile it is.
Work out how many matches there are in the first 40 piles.

Solutions:

a) This sequence is arithmetic. Therefore, \( T_{25} = 10 + (24)(8) = 202 \)
There are 202 matches in the 25th pile.

b) \( T_n = 10 + (n - 1)8 = 98 \)
\( 8n - 8 = 88 \)
\( 8n = 96 \)
\( n = 12 \)

c) \( S_{40} = \frac{40}{2}[2(10) + (40 - 1)8] \)
\( = 6640 \)
Activity 2.9

1. On a certain day, 2 learners are found to be infected with the flu virus. Each day thereafter, the number of new learners infected is three times the previous day’s total.

   a) Calculate how many people become infected on the 5\textsuperscript{th} day.
   b) If there are 2186 learners enrolled at the school, assuming that no one is absent, how long would it take for all the learners to become infected?
   c) Calculate the total number of learners that are infected after 3 days of school.

2. A census is done on a city and it is found that there is a population of 5000 people. Every successive year, the population increases by a tenth of the previous year’s total population.

   a) How many people are there in the third year after the census was completed?
   b) In what year after the census did the population reach 15000?
   c) How many people are found to be living in this city after 10 years?
Answers for Activities

Activity 2.1

1. a) Formula: \( T_n = x^2 - 1 \)
   Next three terms: 24; 35; 48

   b) Formula: terms 1, 3 and 5 are increasing negative odd numbers, terms 2, 4 and 6 are increasing positive odd numbers
   Next three terms: -7; 7; -9.

   c) Formula: multiplying each new term by 2 so: \( T_n = \left( \frac{1}{64} \right) 2^{n-1} \)
   Next three terms: \( \frac{1}{8}; \frac{1}{4}; \frac{1}{2} \)

   d) Formula: Increasing order of \( 3^{n-1} \)
   Next three terms: 81; 243; 729

   e) Formula: adding alternatively either a 1 or 0 after the decimal point.
   Next three terms: 0.10101; 0.010101; 0.1010101

   f) Formula: Multiplying the next term by \( 2x^2y \)
   Next three terms: \( 8x^5y^7; 16x^6y^9; 32x^7y^{11} \)
Activity 2.2

1. a) 2  5  10  17  26
   3  5  7  9
   2  2  2

   \[ a + b + c = 2 \]
   \[ 3a + b = 3 \]
   \[ 2a = 2 \]

   \[ a = 1 \]
   \[ 3(1) + b = 3 \]
   \[ b = 0 \]
   \[ 1 + 0 + c = 2 \]
   \[ c = 1 \]

   \[ T_n = n^2 + 1 \]

b) 1  2  7  16  29  46
   1  5  9  13  17
   4  4  4  4

   \[ a + b + c = 1 \]
   \[ 3a + b = 1 \]
   \[ 2a = 4 \]

   \[ a = 2 \]
   \[ 3(2) + b = 1 \]
   \[ b = -5 \]
   \[ 2 - 5 + c = 1 \]
   \[ c = 4 \]

   \[ T_n = 2n^2 - 5n + 4 \]

c) 3  6  12  21  33
   3  6  9  12
   3  3  3

   \[ a + b + c = 3 \]
   \[ 3 \left(\frac{3}{2}\right) + b = 3 \]
   \[ 3a + b = 3 \]
   \[ b = -\frac{3}{2} \]

   \[ 2a = 3 \]
   \[ \frac{3}{2} + -\frac{3}{2} + c = 3 \]

   \[ a = \frac{3}{2} \]
   \[ c = 3 \]

   \[ T_n = \frac{3}{2} n^2 - \frac{3}{2} n + 3 \]
d) \[\begin{array}{cccccc}
1 & 2 & 3 & 7 & 10 & 12 \\
2 & 5 & 5 & 12 & 17 & 22 \\
5 & 5 & 5 & 5 & 5 & 22 \\
\end{array}\]

\[\begin{align*}
a + b + c &= 1 \\
3a + b &= 2 \\
2a &= 5 \\
a &= \frac{5}{2} \\
\end{align*}\]

\[\begin{align*}
three \left(\frac{5}{2}\right) + b &= 2 \\
b &= -\frac{11}{2} \\
\frac{5}{2} - \frac{11}{2} + c &= 1 \\
c &= 4 \\
T_n &= \frac{5}{2}n^2 - \frac{11}{2}n + 4
\]

e) \[\begin{array}{cccccc}
3 & 2 & 5 & 3 & 8 & 12 \\
2 & 1 & 3 & 1 & 4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}\]

\[\begin{align*}
a + b + c &= 3 \\
3a + b &= 2 \\
2a &= 1 \\
a &= \frac{1}{2} \\
\end{align*}\]

\[\begin{align*}
three \left(\frac{1}{2}\right) + b &= 2 \\
b &= \frac{1}{2} \\
\frac{1}{2} + \frac{1}{2} + c &= 3 \\
c &= 2 \\
T_n &= \frac{1}{2}n^2 + \frac{1}{2}n + 2
\]

f) \[\begin{array}{cccccc}
2 & 4 & 6 & 11 & 17 & 24 \\
1 & 5 & 1 & 6 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}\]

\[\begin{align*}
a + b + c &= 2 \\
3a + b &= 4 \\
2a &= 1 \\
a &= \frac{1}{2} \\
\end{align*}\]

\[\begin{align*}
three \left(\frac{1}{2}\right) + b &= 4 \\
b &= \frac{5}{2} \\
\frac{1}{2} + \frac{5}{2} + c &= 2 \\
c &= -1 \\
\therefore T_n &= \frac{1}{2}n^2 + \frac{5}{2}n - 1
\]

2. \[\begin{array}{cccccc}
3 & 4 & 7 & 12 & 19 & 28 \\
1 & 1 & 2 & 2 & 2 & 2 \\
\end{array}\]

a) \[\begin{array}{cccccc}
9 & 28 & 11 & 39 & 13 & 52 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}\]
b) \( a + b + c = 3 \)  
\[ \begin{align*}
3a + b &= 1 \\
2a &= 2 \\
a &= 1
\end{align*} \]
\[ \begin{align*}
b &= -2 \\
1 - 2 + c &= 3 \\
c &= 4
\end{align*} \]
\( T_n = n^2 - 2n + 4 \)

c) \( T_{22} = (22)^2 - 2(22) + 4 \)
\[ = 444 \]

d) \( T_n = 199 = n^2 - 2n + 4 \)
\[ 0 = n^2 - 2n - 195 \]
\[ 0 = (n - 15)(n + 13) \]
\( n = 15 \) or \( n = -13 \)  
[To find the factors go to the table mode on your calculator (press MODE 3)]
then type in:
\[ \begin{align*}
\text{N/A} \\
\text{ENTER} \\
\text{ENTER} \\
\text{ENTER} \
\end{align*} \]
Press the button and look at the factor pairs for example, does 1 and -195 add up to -2? ... No, so we look at the next pair. We continue doing this until we have found both factors, in our example 13 and -15.
Once we have our factors we can put them back into the brackets.

3. \[
\begin{array}{ccccccc}
2 & 4 & 5 & 9 & 8 & 17 \\
3 & & & & & \\
\end{array}
\]

a) \[
\begin{array}{ccccccc}
11 & 28 & 42 & 17 & \\
3 & 3 & & & 3 & \\
\end{array}
\]

b) \( a + b + c = 2 \)
\[ \begin{align*}
3a + b &= 2 \\
2a &= 3 \\
a &= \frac{3}{2}
\end{align*} \]
\[ \begin{align*}
b &= -\frac{5}{2} \\
\frac{3}{2} - \frac{5}{2} + c &= 2 \\
c &= 3
\end{align*} \]
\[ T_n = \frac{3}{2}n^2 - \frac{5}{2}n + 3 \]

c) \( T_{13} = \frac{3}{2}(13)^2 - \frac{5}{2}(13) + 3 \)
\[ = 224 \]
d) $T_n = 612 = \frac{3}{2}n^2 - \frac{5}{2}n + 3$

$1224 = 3n^2 - 5n + 6$

$0 = 3n^2 - 5n - 1218$

$0 = (3n + 58)(n - 21)$

$n = -\frac{58}{3}$ or $n = 21$

N/A $\Rightarrow$ because n cannot be negative or a fraction.

4.  

a) Before you can find $n$, you need to find the formula:

$a + b + c = 10$

$3(3) + b = 10$

$3a + b = 10$

$b = 1$

$2a = 6$

$3 + 1 + c = 10$

$a = 3$

$c = 6$

$T_n = 3n^2 + n + 6$

$206 = 3n^2 + n + 6$

$0 = 3n^2 + n - 200$

$0 = (3n + 25)(n - 8)$

$n = -\frac{25}{3}$ or $n = 8$

N/A

b) For this question, continue the sequence until your last term (206)

Now add the terms:

$10 + 20 + 36 + 58 + 86 + 120 + 160 + 206$

= 696 chickens

2 weeks = 14 days $n = 14$

$T_{14} = 3(14)^2 + (14) + 6$

= 608 chickens on the 14th day.
Activity 2.3

1.

a) \(-2; -1; 0; 1; 2; \ldots \) \(T_{100} = 97\)

b) \(6; 5; 4; 3; 2; \ldots \) \(T_{100} = -93\)

c) \(-2; 0; 2; 4; 6; \ldots \) \(T_{100} = 196\)

d) \(4; 7; 10; 13; 16; \ldots \) \(T_{100} = 301\)

e) \(1; 0; -1; -2; -3; \ldots \) \(T_{100} = -98\)

f) \(\frac{4}{3}; \frac{5}{3}; 5; \frac{5}{3}; \frac{5}{3}; \ldots \) \(T_{100} = 37 \frac{1}{3}\)

2.

a) \(T_{12} = 4(12) + 1 = 49\)

b) \(T_{4} = 3(4) + \frac{1}{2} = 12 \frac{1}{2}\)

c) \(T_{50} = \frac{2(50)+2}{50} = \frac{102}{50} = \frac{51}{25}\)

You can use the table function on the SHARP EL-W535HT calculator to find the terms in the pattern.

To do this you press \(\text{MODE} \ 3\) then it will ask you “function?” you enter the \(T_n\) formula into the calculator using \(x\) in place of \(n\). For example in 1a. you would press \(\text{ALPHA} \ RCL \ x \ - \ 3 \ = \ \ \text{ENTER}^1\) the calculator will then ask you start? You press \(1 \ \ \text{ENTER}^1\) then step - you press \(1 \ \ \text{ENTER}^1\) the calculator will then give you a table with X and ANS. X is your term position and continues indefinitely and ANS gives you the term value. You can use this function with any of the patterns and you can find any term position by scrolling down the table - press the \(\downarrow\) key. You can also enter the position you are looking for by putting as your start.

You can find the answer to this by putting the substitution straight into your EL-W535HT by pressing \(4 \ \times \ 1 \ 2 \ \ + \ 1 \ \ \text{ENTER}^1\)

To put this into your EL-W535HT you press \(\text{2ndF} \ 2 \ ( \ 5 \ 0 \ ) \ +\) if you want the improper fraction simply press the \(\text{frac}\) button, if you press the button again you will have your answer in decimal form.
3. a) 9; 12; 15…

\[ T_n = a + (n - 1)d \]
\[ = 9 + (n - 1)(3) \]
\[ = 9 + 3n - 3 \]
\[ T_n = 3n + 6 \]
\[ \therefore T_{15} = 3(15) + 6 \]
\[ = 51 \]
\[ \therefore T_{100} = 3(100) + 6 \]
\[ = 306 \]

b) 0; \ -\frac{5}{2}; \ -5 …. 

\[ T_n = 0 + (n - 1)\left(-\frac{5}{2}\right) \]
\[ \therefore T_n = -\frac{5}{2}n + \frac{5}{2} \]
\[ \therefore T_{15} = -\frac{5}{2}(15) + \frac{5}{2} \]
\[ = -35 \]
\[ \therefore T_{100} = -\frac{5}{2}(100) + \frac{5}{2} \]
\[ = -247 \frac{1}{2} \]

c) 7 + 9x; 8 + 11x; 9 + 13x ...

\[ T_n = 7 + 9x + (n - 1)(1 + 2x) \]
\[ = 7 + 9x + n + 2xn + 1 - 2x \]
\[ \therefore T_n = n + 2xn + 7x + 8 \]
\[ \therefore T_{15} = 15 + 2x(15) + 7x + 8 \]
\[ = 15 + 30x + 7x + 8 \]
\[ = 37x + 23 \]
\[ \therefore T_{100} = 100 + 2x(100) + 7x + 8 \]
\[ = 100 + 200x + 7x + 8 \]
\[ = 207x + 108 \]

d) \ a = 5; \ d = 7x 

\[ T_n = 5 + (n - 1)(7x) \]
\[ = 5 + 7xn - 7x \]
\[ \therefore T_n = 7xn - 7x + 5 \]
\[ \therefore T_{15} = 7x(15) - 7x + 5 \]
\[ = 105x - 7x + 5 \]
\[ = 98x + 5 \]
\[ \therefore T_{100} = 7x(100) - 7x + 5 \]
\[ = 700x - 7x + 5 \]
\[ = 693x + 5 \]

4. a) -2; 1; 4;… (109) 

\[ \therefore 109 = -2 + (n - 1)(3) \]
\[ \therefore 111 = 3(n - 1) \]
\[ \therefore 37 = n - 1 \]
\[ \therefore n = 38 \]

b) \ \frac{1}{2}; \ -\frac{3}{7}; \ -\frac{9}{14}; \… (-19) 

\[ \therefore -19 = \frac{1}{2} + (n - 1)\left(-\frac{13}{14}\right) \]
\[ \therefore -\frac{39}{2} = \left(-\frac{13}{14}\right)(n - 1) \]
\[ \therefore 21 = n - 1 \]
\[ \therefore n = 22 \]
c) \[x; \quad 2x + 3; \quad 3x + 6; \ldots (50x + 147)\]

\[
\therefore 50x + 147 = x + (n - 1)(x + 3) \\
\therefore 49x + 147 = (x + 3)(n - 1) \\
\therefore \frac{49(x+3)}{(x+3)} = n - 1 \\
\therefore n = 50
\]

d) \[
\log \sqrt{2}; \quad \log 4; \quad \log 8\sqrt{2} \ldots \quad (\log 2^{23})
\]

\[
\therefore \log 2^{\frac{1}{2}}; \quad \log 2; \quad \log 2^{\frac{3}{2}} \ldots \quad (\log 2^{23})
\]

\[
\therefore \frac{1}{2}\log 2; \quad 2\log 2; \quad 3\frac{1}{2}\log 2 \ldots \quad (23\log 2)
\]

\[
\therefore 23\log 2 = \frac{1}{2}\log 2 + (n - 1)\left(\frac{3}{2}\log 2\right) \\
\therefore 22\frac{1}{2}\log 2 = \left(\frac{3}{2}\log 2\right)(n - 1) \\
\therefore 15 = n - 1 \\
\therefore n = 16
\]

5. a) \[T_4 = 14; \quad T_{20} = 94\]

\[
\therefore a + 19d = 94 \\
- a + -3d = -14 \\
\hline
16d = 80 \\
\therefore d = 5 \\
\therefore T_n = -1 + (n - 1)(5) \\
\therefore T_{14} = 5(14) - 6 \\
\therefore a = -1 \\
\therefore -1; \quad 4; \quad 9 \ldots
\]

b) \[T_7 = 12; \quad T_{33} = -40\]

\[
\therefore a + 32d = -40 \\
- a + -6d = -12 \\
\hline
26d = -52 \\
\therefore d = -2 \\
\therefore T_n = 24 + (n - 1)(-2) \\
\therefore T_{14} = -2(14) + 26 \\
\therefore a = 24 \\
\therefore 24; \quad 22; \quad 20; \ldots\]
c) \( T_5 = 2 + 3x \); \( d = 1 + x \)

\[ \therefore a + 4(1 + x) = 2 + 3x \quad \therefore T_n = -2 - x + (n - 1)(1 + x) \]
\[ \therefore a + 4 + 4x = 2 + 3x \quad \therefore T_n = -2 - x + n + nx - 1 - x \]
\[ \therefore a = -2 - x \quad \therefore T_n = -3 - 2x + n + xn \]

\[ \therefore -2 - x; \ -1; \ x... \quad \therefore T_{14} = -3 - 2x + 14 + 14x \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = 11 + 12x \]

d) \( T_6 = 5x - 2; \quad a = -3 \)

\[ \therefore -3 + 5(d) = 5x - 2 \quad \therefore T_n = -3 + (n - 1)\left(x + \frac{1}{5}\right) \]
\[ \therefore 5d = 5x + 1 \quad \therefore T_n = -3 + xn - x + \frac{1}{5}n - \frac{1}{5} \]
\[ \therefore d = x + \frac{1}{5} \quad \therefore T_n = -3\frac{1}{5} + xn - x + \frac{1}{5}n \]

\[ \therefore -3; \ -2\frac{4}{5} + x; \ -2\frac{3}{5} + 2x; ... \]
\[ \therefore T_{14} = -3\frac{1}{5} + xn - x + \frac{1}{5}n \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = -\frac{2}{5} + 13x \]

6. \( x + 3; \ 2x - 2; \ 5x + 1; ... \)

a) \( 2x - 2 - (x + 3) = 5x + 1 - (2x - 2) \)
\( 2x - 2 - x - 3 = 5x + 1 - 2x + 2 \)
\( x - 5 = 3x + 3 \)
\[ \therefore -2x = 8 \]
\[ \therefore x = -4 \]

b) \( -4 + 3; \ 2(-4) - 2; \ 5(-4) + 1 \)
\[ \therefore -1; \ -10; \ -19 \]

c) \( \therefore T_n = -1 + (n - 1)(-9) \)
\( \therefore T_{20} = -1 + (20 - 1)(-9) \)
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = -172 \]

d) \( -64 = -1 + (n - 1)(-9) \)
\( -63 = -9(n - 1) \)
\[ \therefore n - 1 = 7 \]
\[ \therefore n = 8 \]
Activity 2.4

1. 
   a) \( T_{20} = 6 + (19) \cdot 6 = 120 \)
   b) \( T_{20} = -11 + (19) \cdot 2 = 27 \)
   c) \( T_{20} = (p + 2q) + (19)(2p + q) \)
      \[ = p + 2q + 38p + 19q \]
      \[ = 39p + 21q \]
   d) \( T_{20} = -\frac{1}{4} + (19) \cdot \frac{3}{4} \)
      \[ = \frac{-56}{4} \]
      \[ = 14 \]
   e) (i) \( T_6 = 2 + 5d \)
      \[ 62 = 2 + 5d \]
      \[ 60 = 5d \]
      \[ 12 = d \]
   (ii) \( T_{20} = 2 + (19) \cdot 12 \)
      \[ = 230 \]

2.a) \( T_n = 4n - 2 \)
   (i) \( T_1 = 4(1) - 2 = 2 \)
   a = 2
   (ii) \( T_1 = 2 \)
   \( T_2 = 4(2) - 2 = 6 \)
   \( T_3 = 4(3) - 2 = 10 \)
   d = 4

b) \( T_n = 5 + 3n \)
   (i) \( T_1 = 5 + 3(1) = 8 \)
   a = 8
   (ii) \( T_1 = 8 \)
   \( T_2 = 5 + 3(2) = 11 \)
   \( T_3 = 5 + 3(3) = 14 \)
   d = 3

c) \( T_n = \frac{1}{2}n \)
   (i) \( T_1 = \frac{1}{2}(1) = \frac{1}{2} \)
   a = \( \frac{1}{2} \)
   (ii) \( T_1 = \frac{1}{2} \)
   \( T_2 = \frac{1}{2} (2) = 1 \)
   \( T_3 = \frac{1}{2} (3) = \frac{3}{2} \)
   d = \( \frac{1}{2} \)

d) \( T_n = 6 - 2n \)
   (i) \( T_1 = 6 - 2(1) = 4 \)
   a = 4
   (ii) \( T_1 = 4 \)
   \( T_2 = 6 - 2(2) = 2 \)
   \( T_3 = 6 - 2(3) = 0 \)
   d = -2
3. 

a) \( a = 2, \ d = 2 \)

\[ T_n = 2 + (n - 1).2 \]
\[ 48 = 2 + 2n - 2 \]
\[ 48 = 2n \]
\[ 24 = n \]
\[ T_{24} = 48 \]

b) \( a = -3, \ d = 4 \)

\[ T_n = -3 + (n - 1).4 \]
\[ 81 = -3 + 4n - 4 \]
\[ 88 = 4n \]
\[ 22 = n \]
\[ T_{22} = 81 \]

c) \( a = 2\frac{1}{4}, \ d = \frac{1}{4} \)

\[ T_n = 2\frac{1}{4} + (n - 1) \cdot \left( \frac{1}{4} \right) \]
\[ 6 = 2\frac{1}{4} + \frac{1}{4}n - \frac{1}{4} \]
\[ 4 = \frac{1}{n} \]
\[ \frac{19}{4} = \frac{1}{n} \]
\[ 19 = n (x4) \]
\[ T_{19} = 6\frac{3}{4} \]

d) \( a = a, \ d = (a + 1) \)

\[ T_n = a + (n - 1) (a + 1) \]
\[ 12a + 11 = a + an + n - a - 1 \]
\[ 12a + 12 = an + n \]
\[ 12(a + 1) = n(a + 1) \]
\[ 12 = n [(a + 1)] \]
\[ T_{12} = 12a + 11 \]

4. 

a) \( T_8 = a + 7d = 22 \) (1)

\[ T_{17} = a + 16d = 49 \) (2)

Subtract (1) from (2):
\[ 9d = 27 \]
\[ d = 3 \]

Substitute \( d = 3 \), into (1):
\[ 22 = a + 7(3) \]
\[ 1 = a \]
\[ T_1 = 1, \ T_2 = 4, \ T_3 = 7 \]
\[ T_{25} = 1 + 24.3 = 73 \]

b) \( T_{11} = a + 10d = -28 \) (1)

\[ T_{30} = a + 29d = -104 \) (2)

Subtract (1) from (2):
\[ 19d = -76 \]
\[ d = -4 \]

Substitute \( d = -4 \), into (1):
\[ -28 = a + 10(-4) \]
\[ 12 = a \]
\[ T_1 = 12, \ T_2 = 8, \ T_3 = 4 \]
\[ T_{25} = 12 + 24 \cdot -4 = -84 \]

c) \( T_6 = a + 5d = 10 \) (1)

\[ T_{21} = a + 20d = 14\frac{1}{2} \) (2)

Subtract (1) from (2):
\[ 15d = \frac{9}{2} \]
\[ d = \frac{3}{10} \]

Substitute \( d = \frac{3}{10} \) into (1):
\[ 10 = a + 5\left( \frac{3}{10} \right) \]
\[ 10 - \frac{3}{2} = a \]

d) \( T_4 = a + 3d = 7x + 10 \) (1)

\[ T_{15} = a + 14d = 29x + 43 \) (2)

Subtract (1) from (2):
\[ 11d = 22x + 33 \]
\[ 11d = 11(2x + 3) \]
\[ d = 2x + 3 \]

Substitute \( d = 2x + 3 \) into (1):
\[ 7x + 10 = a + 3(2x + 3) \]
\[ \frac{17}{2} = a \]
\[ T_1 = \frac{17}{2}, \; T_2 = \frac{44}{5}, \; T_3 = \frac{91}{10} \]
\[ T_{25} = \frac{17}{2} + 24 \cdot \left( \frac{3}{10} \right) = \frac{157}{10} \]
\[ 7x + 10 = a + 6x + 9 \]
\[ x + 1 = a \]

\[ T_1 = x + 1, \; T_2 = 3x + 4, \; T_3 = 5x + 7 \]
\[ T_{25} = x + 1 + 24(2x + 3) = x + 1 + 48x + 72 = 49x + 73 \]

**Activity 2.5**

1. 
   a) \[ a = \frac{1}{4} \text{ and } \frac{T_2}{T_1} = \frac{2}{5} \text{ and } \frac{T_3}{T_2} = \frac{2}{5} \] therefore this is a geometric sequence as there is a common ratio of \( \frac{2}{5} \).
   \[ T_4 = \frac{4}{625}; \; T_5 = \frac{8}{3125} \]
   b) \[ a = 3 \text{ and } \frac{T_2}{T_1} = 4, \; \frac{T_3}{T_2} = 4 \] therefore this is a geometric sequence as there is a common ratio of 4.
   \[ T_4 = 192, \; T_5 = 768, \; T_6 = 3072 \]
   c) \[ a = -7 \text{ and } T_2 - T_1 = -3, \; T_3 - T_2 = -3 \] therefore this is an arithmetic sequence as there is a common difference of -3.
   \[ T_4 = -16, \; T_5 = -19, \; T_6 = -22 \]
   d) \[ a = 5 \text{ and } T_2 - T_1 = 4, \; T_3 - T_2 = 4 \] therefore this is an arithmetic sequence as there is a common difference of 4.
   \[ T_4 = 17, \; T_5 = 21, \; T_6 = 25 \]
   e) \[ a = -\frac{3}{4} \text{ and } \frac{T_2}{T_1} = \frac{3}{2}, \; \frac{T_3}{T_2} = \frac{3}{2} \] therefore this is a geometric sequence as there is a common ratio of \( \frac{3}{2} \).
   \[ T_4 = -\frac{81}{32}; \; T_5 = -\frac{243}{64}, \; T_6 = -\frac{729}{128} \]
   f) \[ a = \frac{2}{7} \text{ and } T_2 - T_1 = \frac{1}{2}, \; T_3 - T_2 = \frac{1}{2} \] therefore this is an arithmetic sequence as there is a common difference of \( \frac{1}{2} \).
   \[ T_4 = \frac{25}{14}; \; T_5 = \frac{16}{7}; \; T_6 = \frac{39}{14} \]
2.

a) \( T_{12} = ar^{11} = \frac{1}{1024} \) 
\[ 2r^{11} = \frac{1}{1024} \]
\[ r^{11} = \frac{1}{2048} \]
\[ r = \frac{1}{11\sqrt{2048}} \]
\[ r = -3 \]

b) \( T_{12} = ar^{11} = -88573 \frac{1}{2} \) 
\[ \frac{1}{2}r^{11} = \frac{-177147}{2} \]
\[ r^{11} = -177147 \]
\[ r = \frac{-11\sqrt{-177147}}{2048} \]
\[ r = -3 \]

\( T_{21} = 2 \times \left(\frac{1}{2}\right)^{20} \)
\[ T_{21} = \frac{1}{524288} \]

\( T_{21} = 1743392201 \)

Again, you can plug this straight into your SHARP EL-W535HT calculator by pressing:

\[ 2 \ 1 \ 2 \ 0 \ 4 \ \text{ ENTER} \]

Again, you can plug this straight into your SHARP EL-W535HT calculator by pressing:

\[ 2 \ ( \ 1 \ 2 \ 0 \ ) \ \text{ ENTER} \]

\( T_{21} = 2 \times \left(\frac{1}{2}\right)^{20} \)
\[ T_{21} = \frac{1}{524288} \]

\( T_{21} = 1743392201 \)

\( T_{21} = 8 \times \left(\frac{3}{2}\right)^{20} \)
\[ T_{21} = 26602.05384 \]

c) \( T_{12} = ar^{11} = 8192x^{12} \)
\[ 4xr^{11} = 8192x^{12} \]
\[ r^{11} = 2048x^{11} \]
\[ \frac{11\sqrt{r^{11}}}{2048} = \frac{11\sqrt{2048}x^{11}}{2048} \]
\[ r = 2x \]

\[ T_{21} = 4x(2x)^{20} \]
\[ T_{21} = 4194304x^{21} \]

\( T_{21} = 2048 \times \left(\frac{3}{2}\right)^{20} \)
3.  
\[ T_n = ar^{n-1} \]
\[ 1 \cdot (2)^{n-1} = 32768 \]
\[ \log_2 32768 = n - 1 \]
\[ n - 1 = 15 \]
\[ n = 16 \]

Remember from logs that you can change an exponential equation into a log equation in order to find the exponential unknown. Press the following sequence of keys to get the answer:

\[ \boxed{2 \ 8 \ \text{ENTER}} \]

\[ \boxed{6 \ \text{ENTER}} \]

\[ \boxed{2 \ 3 \ 2 \ 7} \]

\[ c) \quad T_n = ar^{n-1} \]
\[ 4096 = \frac{1}{16} (2)^{n-1} \]
\[ 65536 = (2)^{n-1} \]
\[ \log_2 65536 = n - 1 \]
\[ n - 1 = 16 \]
\[ n = 17 \]

\[ d) \quad T_n = ar^{n-1} \]
\[ \frac{1}{32} = 8(\frac{1}{4})^{n-1} \]
\[ \frac{1}{256} = (\frac{1}{4})^{n-1} \]
\[ \log_{\frac{1}{4}} \frac{1}{256} = n - 1 \]
\[ n - 1 = 4 \]
\[ n = 5 \]

4.  
\[ a) \quad T_n = ar^{n-1} \]
\[ T_5 = 2r^4 = 32 \]
\[ r^4 = 16 \]
\[ \sqrt[4]{r^4} = \sqrt[4]{16} \]
\[ r = 2 \]

\[ c) \quad T_n = ar^{n-1} \]
\[ T_6 = \frac{1}{3} r^5 = 2592 \]
\[ r^5 = 2592 \]
\[ \sqrt[5]{r^5} = \sqrt[5]{2592} \]
\[ \therefore r = 6 \]

\[ b) \quad T_n = ar^{n-1} \]
\[ T_7 = r^6 = 729 \]
\[ \sqrt[6]{r^6} = \sqrt[6]{729} \]
\[ r = 3 \]

\[ d) \quad T_n = ar^{n-1} \]
\[ T_9 = -5r^8 = -\frac{5}{256} \]
\[ r^8 = \frac{1}{256} \]
\[ \sqrt[8]{r^8} = \sqrt[8]{\frac{1}{256}} \]
\[ \therefore r = \pm \frac{1}{2} \]
5. \[ T_1 = (p + 2) \]
\[ T_2 = (p - 2) \]
\[ T_3 = p \]
\[
\frac{T_2}{T_1} = \frac{\frac{p-2}{p+2}}{\frac{p}{p-2}} = \frac{p}{p-2}
\]
\[ p(p + 2) = (p - 2)(p - 2) \]
\[ p^2 + 2p = p^2 - 4p + 4 \]
\[ 6p = 4 \]
\[ p = \frac{2}{3} = \text{term 3} \]

\[ \therefore T_1 = p + 2 \]
\[ \because T_2 = p - 2 \]
\[ = \frac{2}{3} + 2 \]
\[ = \frac{8}{3} \]

\[ \therefore T_1 = p + 2 \]

Activity 2.6

1.

a) \[ T_n = -2n \rightarrow T_1 = -2(1)T_2 = -2(2) \]
\[ = -2 \]
\[ = -4 \]
\[ = -6 \]

\[ \therefore S_3 = \frac{3}{2}[2(-2) + (3 - 1)(-2)] \]
\[ = \frac{3}{2}[-8] \]
\[ = -12 \]

b) \[ T_n = 4 - n \rightarrow T_1 = 4 - 1 \]
\[ = 3 \]
\[ T_2 = 4 - 2 \]
\[ = 2 \]
\[ T_3 = 4 - 3 \]
\[ = 1 \]

\[ \therefore S_3 = \frac{3}{2}[2(3) + (3 - 1)(-1)] \]
\[ = \frac{3}{2}[4] \]
\[ = 6 \]

2. \[ T_n = 2n + 3 \]
\[ \therefore T_1 = 2(1) + 3 \]
\[ = 5 \]
\[ T_2 = 2(2) + 3 \]
\[ = 7 \]
\[ T_3 = 2(3) + 3 = 9 \]

\[ \therefore a = 5 \text{ and } d = 2 \]
\[ S_{10} = \frac{10}{2}[2(5) + (10 - 1)(2)] \]
\[ = 140 \]

3. \[ S_{10} = 120 \]
\[ S_{11} = 144 \]
\[ \therefore \text{the value of } T_{11} \text{ is} \]
\[ S_{11} - S_{10} = 144 - 120 \]
\[ = 24 \]
4. \[ S_n = 3n^2 + n \]

\[ S_1 = 3(1)^2 + 1 = 4 \quad T_1 = S_1 = 4 \]
\[ S_2 = 3(2)^2 + 2 = 14 \quad T_2 = S_2 - S_1 = 14 - 4 = 10 \]
\[ S_3 = 3(3)^2 + 3 = 30 \quad T_3 = S_3 - S_2 = 30 - 14 = 16 \]
\[ S_4 = 3(4)^2 + 4 = 52 \quad T_4 = S_4 - S_3 = 52 - 30 = 22 \]
\[ S_5 = 3(5)^2 + 5 = 80 \quad T_5 = S_5 - S_4 = 80 - 52 = 28 \]

There is a common difference of 6, therefore this is an arithmetic sequence.

5. 
   a) \[ a = 3, \quad d = 3 \quad T_n = 36 = 3 + (n - 1)(3) \]
   \[ 36 = 3 + 3n - 3 \]
   \[ 3n = 36 \]
   \[ \therefore n = 12 \]

   \[ S_{12} = \frac{12}{2} [2(3) + (12 - 1)(3)] \]
   \[ = 6[39] \]
   \[ = 234 \]

   b) \[ a = 2, \quad d = \frac{1}{2} \quad T_n = 10 = 2 + (n - 1)\left(\frac{1}{2}\right) \]
   \[ 10 = 2 + \frac{1}{2} n - \frac{1}{2} \]
   \[ n = \frac{17}{2} \quad \therefore n = 17 \]

   \[ S_{17} = \frac{17}{2} \left[ 2(2) + (17 - 1)\left(\frac{1}{2}\right) \right] \]
   \[ = \frac{17}{2} [12] \]
   \[ = 102 \]

   c) \[ a = 7, \quad d = -\frac{1}{4} \quad T_n = 3.75 = 7 + (n - 1)\left(-\frac{1}{4}\right) \]
   \[ 3.75 = 7 - \frac{1}{4} n + \frac{1}{4} \]
   \[ \frac{15}{4} = 29 - n \]
   \[ \frac{15}{4} = \frac{29 - n}{4} \]
   \[ -\frac{14}{4} = -\frac{n}{4} \]
   \[ n = 14 \]

   \[ S_{14} = \frac{14}{2} \left[ 2(7) + (14 - 1)\left(-\frac{1}{4}\right) \right] \]
   \[ = 7 \left[ 10\frac{3}{4} \right] \]
   \[ = 75\frac{1}{4} \]
6. \( S_n = 45, \quad T_2 = \frac{3}{2} \) and \( T_5 = 3 \)

\[
\therefore \frac{3}{2} = a + d \quad \ldots \quad 1 \\
3 = a + 4d \quad \ldots \quad 2
\]

Equation 2 – Equation 1:
\[
\frac{3}{2} = 3d \\
\therefore d = \frac{1}{2}
\]

Substitute into Equation 1

\[
\therefore \frac{3}{2} = a + \frac{1}{2} \\
\therefore a = 1
\]

\[
\therefore S_n = 45 = \frac{n}{2} \left[ 2(1) + (n - 1) \left( \frac{1}{2} \right) \right] \\
90 = n \left[ 2 + \frac{1}{2}n - \frac{1}{2} \right] \\
90 = 2n + \frac{1}{2}n^2 - \frac{1}{2}n \\
\therefore 0 = n^2 + 3n - 180 \\
\therefore 0 = (n + 15)(n - 12) \\
\therefore n = -15 \quad \text{or} \quad n = 12
\]

A term position can never be negative \( \therefore n = 12 \)

7. \( T_8 - T_5 = 15 \)

\( \therefore a + 7d - (a + 4d) = 15 \)

\( \therefore 3d = 15 \)

\( \therefore d = 5 \)

\[
\therefore S_6 = 159 = \frac{6}{2} [2a + (6 - 1)(5)] \\
159 = 3[2a + 25] \\
53 = 2a + 25 \\
2a = 28 \\
\therefore a = 14 \\
\therefore S_{10} = \frac{10}{2} [2(14) + (10 - 1)(5)] \\
\therefore S_{10} = 5[73] \\
\therefore S_{10} = 365
\]

8.

a) \( S_{10} = \frac{8 \left( \frac{1}{2} \right)^{10} - 1}{\frac{1}{2} - 1} \)

\[
= 15 \frac{63}{64}
\]
b) \[ S_6 = \frac{2((-2)^6-1)}{-2-1} = \frac{126}{-3} = -42 \]

\[ S_8 = \frac{\frac{1}{3}((\frac{1}{3})^8-1)}{\frac{1}{3}-1} = \frac{3280}{6561} \]

9.

a) \[ T_1 = 3 \quad \text{and} \quad T_7 = \frac{3}{64} \]
\[ \therefore a = 3 \quad \text{Substitute into:} \quad T_7 = \frac{3}{64} = ar^6 \]
\[ 3r^6 = \frac{3}{64} \]
\[ \therefore r^6 = \frac{1}{64} \]
\[ \sqrt[6]{r^6} = \sqrt[6]{\frac{1}{64}} \quad \therefore r = \pm \frac{1}{2} \]
\[ S_{10} = \frac{3\left((\frac{1}{2})^{10}-1\right)}{\frac{1}{2}-1} = 5 \frac{509}{512} \quad \text{OR} \quad S_{10} = \frac{3\left((\frac{1}{2})^{10}-1\right)}{\frac{1}{2}-1} = 1 \frac{511}{512} \]

b) \[ T_1 = 2, \quad \text{and} \quad T_8 = 4374 \]
\[ \therefore a = 2 \quad \text{Substitute into:} \quad T_8 = 4374 = ar^7 \]
\[ 4374 = 2r^7 \]
\[ \therefore r^7 = 2187 \]
\[ \sqrt[7]{r^7} = \sqrt[7]{2187} \quad \therefore r = 3 \]
\[ S_{15} = \frac{2(3^{15}-1)}{3-1} = 14348906 \]

10. \[ S_6 = 504 = \frac{a(2^6-1)}{2-1} \]
\[ 504 = 63a \]
\[ \therefore a = 8 \]
\[ \therefore T_1 = 8 \]
Activity 2.7

1. 
   a) Geometric sequence: \( a = -2, r = 3. \)
      
      \[ S_8 = \frac{-2(2^8-1)}{2-1} \]
      
      \[ \therefore S_8 = -6 \, 560 \]

   b) Arithmetic sequence: \( a = 3, d = 3. \)
      
      \[ S_8 = \frac{8}{2} [2(3) + (8 - 1)(3)] \]
      
      \[ \therefore S_8 = 4[27] \]
      
      \[ \therefore S_8 = 108 \]

   c) Geometric sequence: \( a = , r = 2. \)
      
      \[ S_8 = \frac{\frac{1}{2}(2^8-1)}{2-1} \]
      
      \[ \therefore S_8 = 127 \frac{1}{2} \]

   d) Arithmetic sequence: \( a = 16, d = -4. \)
      
      \[ S_8 = \frac{8}{2} [2(16) + (8 - 1)(-4)] \]
      
      \[ \therefore S_8 = 4[4] \]
      
      \[ \therefore S_8 = 16 \]

2. 
   a) \( \frac{4920}{2} = \frac{\frac{1}{2}(3^n-1)}{3-1} \)
      
      \[ 9841 = \frac{1}{2}(3^n - 1) \]
      
      \[ 19 \, 682 = 3^n - 1 \]
      
      \[ 19 \, 683 = 3^n \]
      
      \[ \log_3 19 \, 683 = n \]
      
      \[ n = 9 \]

   b) \( 285 = \frac{n}{2} [2(5) + (n - 1)(2)] \)
      
      \[ 570 = n[10 + 2n - 2] \]
      
      \[ 570 = 8n + 2n^2 \]
      
      \[ 0 = n^2 + 4n - 285 \]
      
      \[ 0 = (n + 19)(n - 15) \]
      
      \[ \therefore n = 15 ; n \neq -19 \]

   n can never be negative therefore n is only equal to 15.
c) \[16 \times 400 = \frac{5(3^n - 1)}{3 - 1}\]
\[32 \times 800 = 5(3^n - 1)\]
\[6.560 = 3^n - 1\]
\[6.561 = 3^n\]
\[\log_3 6.561 = n\]
\[n = 8\]

3.

a) \[\sum_{n=1}^{5} \left(\frac{1}{2}\right)(4)^{n-1}\]
\[\therefore S_5 = \frac{1}{2} \cdot \frac{4^5 - 1}{4 - 1}\]
\[\therefore S_5 = 170\frac{1}{2}\]

b) \[\sum_{n=3}^{8} \frac{5n}{2}\]
\[\therefore S_6 = \frac{6}{2} \cdot \left[2 \left(7\frac{7}{2}\right) + (6 - 1) \left(2\frac{1}{2}\right)\right]\]
\[\therefore S_6 = 3 \cdot 27\frac{3}{2}\]
\[\therefore S_6 = 82\frac{1}{2}\]

c) \[\sum_{n=6}^{18} 3 \cdot 2n\]
\[\therefore S_{13} = \frac{13}{2} \cdot [2(36) + (13 - 1)(6)]\]
\[\therefore S_{13} = \frac{13}{2} \cdot [144]\]
\[\therefore S_{13} = 936\]

d) \[\sum_{n=2}^{6} \left(\frac{1}{2}\right)^{n-1}\]
\[\therefore S_5 = \frac{2(\frac{1}{2})^5 - 1}{\frac{1}{2} - 1}\]
\[\therefore S_5 = 3\frac{7}{8}\]

4.

a) \[S_n = \frac{n}{2} \cdot [2a + (n - 1)d]\]
\[\therefore 182 = \frac{n}{2} \cdot [2(8) + (n - 1)(6)]\]
\[\therefore 364 = n[16 + 6n - 6]\]
\[\therefore 0 = 16n + 6n^2 - 6n - 364\]
\[\therefore 0 = 6n^2 + 10n - 364\]
\[\therefore 0 = 3n^2 + 5n - 182\]
\[\therefore \log_2 1024 = n\]
\[\therefore 0 = (3n + 26)(n - 7)\]
\[\therefore n = 10\]
\[\therefore n \neq -\frac{2}{3}\] or \(n = 7\) as it needs to be positive.

n also needs to be a whole number i.e. not a fraction.
5.

a) \[ \sum_{n=1}^{100} (3n - 5) \] Arithmetic series with:
   \[ T_1 = -2, T_2 = 1, T_3 = 4, \text{ therefore } a = -2 \text{ and } d = 3 \]
   \[ S_n = \frac{n}{2} [2a + (n - 1)d] \]
   \[ \therefore S_{100} = \frac{100}{2} [2(-2) + (100 - 1)(3)] \]
   \[ \therefore S_{100} = 50[293] \]
   \[ \therefore S_{100} = 14650 \]

b) \[ \sum_{n=1}^{150} \left( \frac{1}{2} \right) (3)^{n-1} \] Geometric series with:
   \[ T_1 = \frac{1}{2}; T_2 = \frac{3}{2}; T_3 = \frac{9}{2}, \text{ therefore } a = \frac{1}{2} \text{ and } r = 3 \]
   \[ S_n = \frac{a(r^n - 1)}{r - 1} \]
   \[ \therefore S_{150} = \frac{\frac{1}{2}(3^{150} - 1)}{3 - 1} \]
   \[ \therefore S_{150} = 9.25 \times 10^{70} \]

6.

a) \[ a = -1, r = 3, \text{ therefore this is a geometric sequence.} \]
   \[ \sum_{n=1}^{k} (-1)(3)^{n-1} \]

b) \[ a = 2, r = , \text{ therefore this is a geometric sequence.} \]
   \[ \sum_{n=1}^{k} (2) \left( \frac{1}{2} \right)^{n-1} \]

Activity 2.8

1.

a) \[ a = 2, \quad r = \frac{1}{3} \]
   \[ S_\infty = \frac{a}{1 - r} \]
   \[ S_\infty = \frac{2}{1 - \frac{1}{3}} \]
   \[ S_\infty = 3 \]

b) \[ a = 5.25, \quad r = \frac{1}{100} \]
   \[ S_\infty = \frac{a}{1 - r} \]
   \[ S_\infty = \frac{5.25}{1 - \frac{1}{100}} \]
   \[ S_\infty = 5 \frac{10}{33} \]
c) \[ a = 1, \quad r = \frac{1}{3} \]

\[ S_\infty = \frac{\frac{a}{1-r}}{1} = 1 \]

\[ S_\infty = 1 + \frac{1}{2} \]

\[ S_\infty = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \]

\[ S_\infty = \frac{1}{2} \]

d) \[ a = 1, \quad r = -\frac{1}{2} \]

\[ S_\infty = \frac{\frac{a}{1-r}}{1} = \frac{2}{3} \]

\[ S_\infty = \frac{1}{1 - (1/2)} = \frac{2}{1} \]

\[ S_\infty = \frac{2}{3} \]

2. \[ T_1 = \frac{2}{3}, \quad T_2 = \frac{4}{3}, \quad T_3 = \frac{8}{3} \]

\[ \therefore a = \frac{2}{3} \quad \text{and} \quad r = 2 \]

\[ \therefore \text{There is no sum to infinity because } r > 1. \]

3. \[ S_\infty = \frac{\frac{a}{1-r}}{1} = \frac{15}{3} \]

\[ \therefore 1 - r = \frac{3}{15} \]

\[ \therefore r = \frac{4}{5} \]

4. \[ S_\infty = \frac{\frac{a}{1-r}}{1} = \frac{7}{16} \]

\[ \therefore a = \frac{7}{16} \times \frac{7}{8} \]

\[ \therefore a = \frac{49}{128} \]

5. \[ a = 2, \quad r = \frac{3}{5} \]

\[ S_\infty = \frac{\frac{a}{1-r}}{1} = \frac{2}{1 - \frac{3}{5}} \]

\[ \therefore S_\infty = \frac{2}{\frac{2}{5}} \]

\[ \therefore S_\infty = 5 \]

Therefore he can climb 5km before the incline is too steep and he will have to use climbing equipment.

Activity 2.9

1.

a) \[ a = 2, \quad r = 3 \]

\[ T_5 = (2)(3)^4 = 2186 \]

\[ T_5 = 162 \]

b) \[ S_r = \frac{a(r^n-1)}{r-1} \]

\[ 2186 = \frac{2(3^n-1)}{3-1} \]

\[ 2186 = 3^n - 1 \]

\[ 2187 = 3^n \]

\[ \log_3 2187 = n \]

\[ n = 7 \]

\[ S_3 = \frac{2(3^3-1)}{3-1} = 26 \]

learners are affected after 3 days

c) \[ T_3 = (5000)\left(\frac{11}{10}\right)^2 \]

\[ T_3 = 6050 \]

\[ T_3 = (5000)\left(\frac{11}{10}\right)^{10-1} \]

\[ T_{10} = 11789.7 \]

\[ \therefore T_{10} = 11790 \]

You can’t get 0.7 of a person so you will have to round off – remember to round up and not down.
b) \[ T_n = 15000 = (5000) \left( \frac{11}{10} \right)^{n-1} \]
\[ \therefore 3 = \left( \frac{11}{10} \right)^{n-1} \]
\[ \log_{\frac{11}{10}} 3 = n - 1 \]
\[ n - 1 = 11.5 \]
\[ n = 12.5 \]

Check: \[ T_{12} = 5000 \left( \frac{11}{10} \right)^{11} = 14265.58 \]
\[ T_{13} = 5000 \left( \frac{11}{10} \right)^{12} = 15692.14 \]

Therefore the population will reach 15000 in the 12th year.