> DOING MATHEMATICS SHOULD ALWAYS MEAN FINDING PATTERNS AND CRAFTING BEAUTIFUL AND MEANINGFUL EXPLANATIONS
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Grade 10 Maths

Number Patterns

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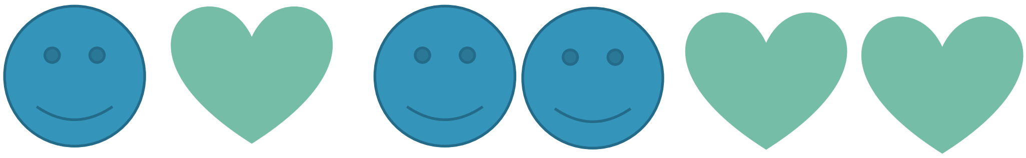
Patterns

Do you remember when you were in primary school and you had to identify the pattern and continue adding the different shapes to the line?

e.g.  ...

Back then the patterns were obvious right? 1 smiley face, 1 heart, 2 smiley faces, 2 hearts, and then we can easily see that the next shapes in the pattern must be 3 smiley faces and then 3 hearts, or one more smiley face than the previous time, and one more heart than the previous time.

Numbers also have patterns. When you are counting in 2's you are adding an extra 2 to your previous number each time: 2 ↘ 2 + 2 ↗ 4 ↘ 4 + 2 ↗ 6 ↘ 6 + 2 ↗ 8 ↘ 8 + 2 ↗ 10... etc...

What happens when you start counting in 2's from a different number, instead of from2?

e.g. 3 ↘ 3 + 2 ↗ 5 ↘ 5 + 2 ↗ 7 ↘ 7 + 2 ↗ 9 ↘ 9 + 2 ↗ 11... and so on.

We are still adding 2 each time, but the pattern looks different, because we started in a different place.

Conjectures and Theories

Recognising the pattern is the first step. Then you need to be able to put what you see into words. A conjecture or theory is a statement about what you believe is happening in the pattern. For the example above, our conjecture (or theory) is that you add 2 to the previous term.

Take a look at the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, and so on...

Can you see the pattern? You are not counting in 2's here. Here the conjecture or theory is that the 2 previous terms are added to get the next term. Can you see it? 1 + nothing is 1; 1 + 1 is 2, 2 + 1 is 3, 3 + 2 is 5 and so it goes on.

There are a couple of common types of patterns that you should be able to recognise.

The first type of pattern is the linear pattern – the counting in something pattern. You keep adding (or subtracting – or in other words, counting backwards) to (or from) a certain number. You add or subtract the same number each time. Here are some examples:

3, 7, 11, 15, 19 ...etc (adding 4 every time)

4, 10, 16, 22, 28 ...etc (adding 6 every time)

91, 88, 85, 82, 79 ...etc (subtracting 3 every time)

The second type of pattern that you need to be able to recognise is the geometric pattern. This pattern occurs when you multiply each previous term (preceding is another word for previous) by the same number. You are not counting in 3's any more, but rather adding to the exponent. Here are some examples:

1, 3, 9, 27, 243 ...etc (multiplying by 3 every time)

3^0, 3^1, 3^2, 3^3, 3^4 ...etc (adding 1 to the exponent each time – more about this in grade 12)

You can also "go backwards" in a geometric pattern. Instead of multiplying each time, you divide by the same number every time (or you can also say that you are multiplying by a fraction each time – remember that fractions are just fancy divides). Here is an example:

32, 16, 8, 4, 2, 1, $\frac{1}{2}$ ...etc.

(You are dividing by 2 or multiplying by $\frac{1}{2}$).

You also have to be able to recognise other mathematical patterns – like the Fibonacci sequence we spoke about earlier, or perfect squares and cubes.

Activity 1

1. Look at the following and determine what is happening in the pattern. Write down the next three numbers in the pattern and then write down your conjecture.

a)	35,	41,	47,	53,	…	b)	48,	24,	12,	6,	…
c)	22,	13,	4,	-5,	…	d)	2,	4,	8,	14,	…
e)	4,	12,	36,	108,	…	f)	101,	121,	141,	161,	…
g)	1,	4,	9,	16,	…	h)	$15\frac{4}{5}$,	$15\frac{1}{5}$,	$14\frac{3}{5}$,	14,	…
i)	3,	6,	11,	18,	…	j)	1,	8,	27,	64,	…

2. Look at the patterns in question 1) and say whether the pattern is linear, geometric or other.

Finding Formulas

In grade 10, you only have to be able to find the formula for a linear pattern (where you add or subtract the same value each time). The easiest way to explain how to find a formula is use an example.

Our pattern is: 5, 8, 11, 14 …

First, we ask ourselves: what do you add (or subtract) each time to get to the next term? In this example we add 3 each time.

Then we ask, what do you need to add to (or subtract from) 3 (what we have added each time – i.e., our constant) to get to 5 (our first term)? The answer is 2.

We can write the formula for this pattern like this: $T_n = 3n + 2$

In this pattern, T_n is our pattern's value at each position n in the pattern.

So, in general our formula looks like this:

Term position what you need to add or subtract to get your first term

$$\boldsymbol{T_n = dn + c}$$

Term value at position n difference (added or subtracted constant)

Each of the variables are explained below in more depth:

- **n** or **term position** is the place or position where the term lies in the pattern or sequence of numbers.
    - In our example 5 is in the first position, so n = 1.
- $\boldsymbol{T_n}$ or **term** is how much that position is worth, or the value of the term at the position n.
    - In our example, our first position in the pattern has a value of 5, the second position has a value of 8, and so on.
- **d** or **difference** is the number that you add or subtract to get to the next term in the pattern.
    - In our example, the difference is +3
- **c** or **what you need** is what you add to or subtract from *dn* to get to the first term's value.
    - In our example, when n = 1 then T = 5 and d = 3, so 3 x 1 = 3. What do we need to add to or subtract from 3 to get to 5? The answer is add 2, in other words c = 2.

Let's try some examples:

Our first pattern is 16, 13, 10, 7, ...

We know that our general formula is $T_n = dn + c$

What is the common difference? -3

So, we have that $T_n = -3n + c$

Now we need to solve for c: 16 = -3(1) + c

Term value difference position of term

16 =-3 + c

19 = c

The formula is $\therefore T_n = -3n + 19$

Our next pattern is -2, 3, 8, 13, ...

Our general formula is $T_n = dn + c$

What is the common difference? +5

So, we have that $T_n = 5n + c$

Now we need to solve for c: -2 = 5(1) + c

Term value difference position of term

-2 =5 + c

-7 = c

The formula is $\therefore T_n = 5n - 7$

You can use your formula to find two things: the value of the term, or, the term's position in the sequence. For the first, you are looking for T_n and for the second, you are looking for the value of n. Simply substitute the values that you have been given and solve for the other value – just like a linear equation.

Here are some examples to help you:

Given the pattern: 7, 5, 3, 1...

Determine (means find) the 12th term.

First, we need to find the general formula:

So: $T_n = dn + c$

$7 = -2(1) + c$ (-2 because we are subtracting 2 each time in the pattern)

$\therefore c = 9$

This means that the formula is $T_n = -2n + 9$

Now we are looking for the value of T when n = 12. This means that we substitute n = 12 into the formula that we found.

$\therefore T_{12} = -2(12) + 9$
$\therefore T_{12} = -24 + 9$
$\therefore T_{12} = -15$

This shows that the 12th term has a value of -15.

Given the pattern: 208, 195, 182, 169...

Determine the position of the term with a value of 26.

First, we need to find the formula for the pattern:

This is: $T_n = dn + c$

$208 = -13(1) + c$ (-13 because we are subtracting 13 each time in the pattern)

$\therefore c = 221$

This means that the formula is $T_n = -13n + 221$

We are looking for the value of n, which means that we need to substitute 26 in the place of T and solve for n:

$\therefore 26 = -13n + 221$
$\therefore 26 - 221 = -13n$
$\therefore -195 = -13n$
$\therefore n = 15$

Thus 26 is position 15 in the pattern.

Remember, in patterns, your n or term position can never be negative.

Activity 2

1. Determine the formula for each of the following patterns:

   a) 3, 7, 11, 15, ...

   b) 5, 11, 17, 23, ...

   c) 25, 12, -1, -14, ...

   d) 8, 38, 68, 98, ...

   e) 18, 30, 42, 54, ...

2. Given the following pattern: 6, 15, 24, 33, ...

   a) Determine the next 3 terms

   b) Write down your conjecture about the pattern

   c) Determine the formula for the pattern

   d) Will the terms in the pattern always be a multiple of 3? Why do you think so?

3. Given the following pattern: 12, 7, 2, -3, ...

   a) Determine the next three terms

   b) Write down your conjecture about the pattern

   c) Determine the formula for the pattern

   d) What is the value of the term in the 10^{th} position

   e) Is -10 a part of the pattern?

4. Susie joins *Tiktok* and finds 6 friends that she knows on the first day. On the second day she adds another 4 friends. If she continues to add 4 friends every day, determine:

   a) The first 5 terms in the pattern

   b) The formula for the number of friends Susie has per day

   c) How many friends will Susie have on the 25^{th} day?

   d) On what day will Susie have 38 friends?

5. Ahmed's mom gives him R210 pocket money. If he spends R30 every day, determine:
   a) How much money he has for the first 4 days
   b) the formula for the pattern
   c) On what day will Ahmed have no more money (hint, when he has R0).
   d) Do you think that Ahmed can continue to spend R30 every day? Why do you think so?

6. Jack and Jill are messaging each other. On the first day Jack sends Jill 12 messages. The next day he sends 28 messages, and the next day he sends 44 messages.
   a) If Jack continues in this pattern, how many messages will he send on the fourth and fifth day?
   b) Determine a formula for the number of messages that Jack sends per day.
   c) How many messages will Jack send Jill on the 14^{th} day?
   d) On which day will Jack send Jill 364 messages?

7. Bobby is a soccer player. He finds that after 1 hour of practice he can get 1 goal in, after 2 hours of practice he scores 3 goals, and after 3 hours of practice he scores 5 goals.
   a) Continue the pattern for 4, 5 and 6 hours of practice
   b) Write down your conjecture
   c) Write down a formula for the above pattern.
   d) How many goals will Bobby score if he practices 17 hours?
   e) How many hours did Bobby spend practicing if he scored 21 goals?
   f) If Bobby needs to score a minimum of 30 goals to get into the team, how many hours would he need to spend practicing?

Answers for the Activities

Activity 1

1\. a) 35, 41, 47, 53, 59, 65, 71

+6 +6 +6 +6 +6 +6

Add 6 to the previous term to get the next term.

b) 48, 24, 12, 6, 3, $1\frac{1}{2}$, $\frac{3}{4}$

÷2 ÷2 ÷2 ÷2 ÷2 ÷2

Divide the previous term by 2 to get the next term.

c) 22, 13, 4, -5, -14, -23, -32

-9 -9 -9 -9 -9 -9

Subtract 9 from the previous term to get the next term.

d) 2, 4, 8, 14, 22, 32, 44

+2 +4 +6 +8 +10 +12

+2 +2 +2 +2 +2

Add two more than you added the previous time to the previous term starting with adding 2. (Or you could say – add consecutive multiples of two (even number).

e) 4, 12, 36, 108, 324, 972, 2916

x3 x3 x3 x3 x3 x3

Multiply the previous term by 3 to get the next term.

f) 101, 121, 141, 161, 181, 201, 221

+20 +20 +20 +20 +20 +20

Add 20 to the previous term.

g) 1, 4, 9, 16, 25, 36, 49

Each term is a perfect square

Find the square of the term position to find the next term.

h) $15\frac{4}{5}$, $15\frac{1}{5}$, $14\frac{3}{5}$, 14, $13\frac{2}{5}$, $12\frac{4}{5}$, $12\frac{1}{5}$

$-\frac{3}{5}$ $-\frac{3}{5}$ $-\frac{3}{5}$ $-\frac{3}{5}$ $-\frac{3}{5}$ $-\frac{3}{5}$

Subtract $\frac{3}{5}$ *from the previous term to find the next term.*

i) 3, 6, 11, 18, 27, 38, 51

Add 2 more than was added to the previous term starting with adding 3 to the first term (or you could say, add consecutive odd numbers).

Or add 2 to the next perfect square in the sequence to the next term

j) 1, 8, 27, 64, 125, 216, 343

Find the cube of the position of each term.

2.

a)	35,	41,	47,	53,	...	*linear*
b)	48,	24,	12,	6,	...	*geometric*
c)	22,	13,	4,	-5,	...	*linear*
d)	2,	4,	8,	14,	...	*other*
e)	4,	12,	36,	108,	...	*geometric*
f)	101,	121,	141,	161,	...	*linear*
g)	1,	4,	9,	16,	...	*other*
h)	$15\frac{4}{5}$,	$15\frac{1}{5}$,	$14\frac{3}{5}$,	14,	...	*linear*
i)	3,	6,	11,	18,	...	*other*
j)	1,	8,	27,	64,	...	*other*

Activity 2

1. Determine the formula for each of the following patterns:

a) 3, 7, 11, 15,
+4 +4 +4

$T_n = dn + b$
$3 = 4(1) + b$
$b = -1$

$\therefore T_n = 4n - 1$

b) 5, 11, 17, 23, …
+6 +6 +6

$T_n = dn + b$
$5 = 6(1) + b$
$b = -1$

$\therefore T_n = 6n - 1$

c) 25, 12, -1, -14, …
-13 -13 -13

$T_n = dn + b$
$25 = -13(1) + b$
$38 = b$

$\therefore T_n = -13n + 38$

d) 8, 38, 68, 98, …
+30 +30 +30

$T_n = dn + b$
$8 = 30(1) + b$
$b = -22$

$\therefore T_n = 30n - 22$

e) 18, 30, 42, 54, …
+12 +12 +12

$T_n = dn + b$
$18 = 12(1) + b$
$b = 6$

$\therefore T_n = 12n + 6$

2. 6, 15, 24, 33, …
+9 +9 +9

a) 42, 51, 60

b) Add 9 to the previous term to get the next term.

c) $T_n = dn + b$

$6 = 9(1) + b$
$b = -3$

$\therefore T_n = 9n - 3$

d) Yes they will be, because the pattern formula is also a multiple of 3. In other words, we can take 3 out 9n -3 as a common factor: $T_n = 3(3n - 1)$

3. 12, 7, 2, -3, ...
-5 -5 -5

a) -8 -13 -18

b) Subtract 5 from the previous term to get the next term.

c) $T_n = dn + b$

$12 = -5(1) + b$

$b = 17$

d) T = -5(10) + 17

T = -50 + 17

T = -33

e) No, it is not. There are 2 ways to prove this:

1. From answer a we can see that -10 is not a part of the sequence.

OR

2. -10 = -5n + 17

-27 = -5n

$5\frac{2}{5}$ = n

Because n is not a whole number, it means that -10 is not a part of the sequence.

4. a) 6 10 14 18 22
+4 +4 +4 +4

b) $T_n = dn + b$

$6 = 4(1) + b$

$2 = b$

$\therefore T_n = 4n + 2$

c) $T_{25} = 4(25) + 2$

$T_{25} = 100 + 2$

$T_{25} = 102$

∴ Susie will have 102 friends on the 25th day.

d) 38 = 4n +2

36 = 4n

n = 9

Susie will have 38 friends on the 9th day.

5. Ahmed's mom gives him R210 pocket money. If he spends R30 every day, determine:

a) 210 180 150 120

b) $T_n = dn + b$

$210 = -30(1) + b$
$240 = b$

$\therefore T_n = -30n + 240$

c) $0 = -30n + 240$

$-240 = -30n$
$n = 8$

$\therefore$ Ahmed will have no money on the 8^{th} day.

d) No, because on the 8^{th} day, he has no money left and you cannot spend negative money.

6. Jack and Jill are messaging each other. On the first day Jack sends Jill 12 messages. The next day he sends 28 messages, and the next day he sends 44 messages.

a) 12 28 44

+16 +16

On the fourth day- 44+ 16 = 60 messages

On the fifth day – 60 + 16 = 76 messages

b) $T_n = dn + b$

$12 = 16(1) + b$
$b = -4$

$\therefore T_n = 16n - 4$

c) $T_{14} = 16(14) - 4$

$T_{14} = 224 - 4$
$T_{14} = 220$

$\therefore$ Jack will send 220 messages on the 14^{th} day.

d) $364 = 16n - 4$

$368 = 16n$
$n = 23$

$\therefore$ Jack sends 364 messages on the 23^{rd} day.

7. a) 1 3 5 7 9 11

+2 +2 +2 +2 +2

b) For every hour more that Bobby practices, he scores 2 more goals.

c) $T_n = dn + b$

$1 = 2(1) + b$
$b = -1$

$\therefore T_n = 2n - 1$

d) $T_{17} = 2(17) - 1$

$T_{17} = 34 - 1$
$T_{17} = 33$

∴ Bobby would score 33 goals.

e) $21 = 2n - 1$

$22 = 2n$
$n = 11$

∴ Bobby spent 11 hours practicing.

f) $30 > 2n - 1$

$31 > 2n$
$15\frac{1}{2} > n$

∴ Bobby would need to practice for at least 16 hours in order to score more than 30 goals.